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NUMERALS OR COUNTERS?

From the *Margarita Philosophica.* (See page 48.)

A Scrap-Book

of

Elementary Mathematics

Notes, Recreations, Essays

By

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PREFACE.

Mathematics is the language of definiteness, the necessary vocabulary of those who know. Hence the intimate connection between mathematics and science.

The tendency to select the problems and illustrations of mathematics mostly from the scientific, commercial and industrial activities of to-day, is one with which the writer is in accord. It may seem that in the following pages puzzles have too largely taken the place of problems. But this is not a text-book. Moreover, amusement is one of the fields of applied mathematics.

The author desires to express obligation to Prof. James M. Taylor, LL.D., of Colgate University (whose pupil the author was for four years and afterward his assistant for two years) for early inspiration and guidance in mathematical study; to many mathematicians who have favored the author with words of encouragement or suggestion while some of the sections of the book have been appearing in periodical form; and to the authors and publishers of books that have been used in preparation. Footnotes give, in most cases, only sufficient reference to identify the book cited. For full bibliographic data see pages 200–205. Special thanks are due to E. B. Escott, M.S., of the mathematics department of the University of Michigan, who read the manuscript. His comments were of especial value in the theory of numbers. Extracts from his notes on that subject (many of them hitherto unpublished) were generously placed at the disposal of the present writer. Where used, mention of the name will generally be found at the place. Grateful acknowledgement is made of the kindness and the critical acumen of Mr. Escott.

The arrangement in more or less distinct sections accounts for occasional repetitions. The author asks the favor of notification of any errors that may be found.

The aim has been to present some of the most interesting and suggestive phases of the subject. To this aim, all others have yielded, except that accuracy has never intentionally been sacrificed. It is hoped that this little book may be found to possess all the unity, completeness and originality that its title claims.

THE AUTHOR.

NEW PALTZ, N. Y., August, 1907.

THE TWO SYSTEMS OF NUMERATION OF LARGE NUMBERS.

What does a billion mean?

In Great Britain and usually in the northern countries of Europe the numeration of numbers is by groups of six figures (10^6 = million, 10^{12} = billion, 10^{18} = trillion, etc.) while in south European countries and in America it is by groups of three figures (10^6 = million, 10^9 = billion, 10^{12} = trillion, etc.). Our names are derived from the English usage: *billion*, the *second* power of a million; *trillion*, the *third* power of a million; etc.

As the difference appears only in such large numbers, which are best written and read by exponents, it is not a matter of practical importance—indeed the difference in usage is rarely noticed—except in the case of *billion*. This word is often heard; and it means a thousand million when spoken by one half of the world, and a million million in the mouths of the other half.

Billion. "A billion does not strike the average mind as a very great number in this day of billion dollar trusts, yet a scientist has computed that at 10:40 a. m., April 29, 1902, only a billion minutes had elapsed since the birth of Christ." One wonders where he obtained the data for such accuracy, but the general correctness of his result is easily verified. "Billion" is here used in the French and American sense (thousand million).

An English professor has computed that if Adam was created in 4004 B. C. (Ussher's chronology), and if he had been able to work 24 hours a day continuously till now at counting at the rate of three a second, he would have but little more than half completed the task of counting a billion in the English sense (million million).

REPEATING PRODUCTS.

If 142857 be multiplied by successive numbers, the figures repeat in the same cyclic order; that is, they read around the circle in the margin

	1	
7		4
5		2
	8	

in the same order, but beginning at a different figure each time.

$$\begin{aligned} 2 \times 142857 &= 285714 \\ 3 \times \quad \text{“} \quad &= 428571 \\ 4 \times \quad \text{“} \quad &= 571428 \\ 5 \times \quad \text{“} \quad &= 714285 \\ 6 \times \quad \text{“} \quad &= 857142 \\ 7 \times \quad \text{“} \quad &= 999999 \\ 8 \times \quad \text{“} \quad &= 1142856. \end{aligned}$$

(When we attempt to put this seven-place number in our six-place circle, the first and last figures occupy the same place. Add them, and we still have the circular order 142857.)

$$\begin{aligned} 9 \times 142857 &= 1285713 \quad (285714) \\ 10 \times \quad \text{“} \quad &= 1428570 \quad (428571) \\ 11 \times \quad \text{“} \quad &= 1571427 \quad (571428) \\ 23 \times \quad \text{“} \quad &= 3285711 \quad (285714) \\ 89 \times \quad \text{“} \quad &= 12714273. \end{aligned}$$

(Again placing in the six-place circular order and adding figures that would occupy the same place, or taking the 12 and adding it to the 73, we have 714285.)

$$356 \times 142857 = 50857092$$

(adding the 50 to the 092, 857142).

The one exception given above ($7 \times 142857 = 999999$) to the circular order furnishes the clew to the identity of this "peculiar" number: 142857 is the repetend of the fraction $\frac{1}{7}$ expressed decimally. Similar properties belong to any "perfect repetend" (repetend the number of whose figures is just one less than the denominator of the common fraction to which the circulate is equal). Thus $\frac{1}{17} = .\dot{0}58823529411764\dot{7}$; $2 \times 0588\cdots = 1176470588235294$ (same circular order); $7 \times 0588\cdots = 4117647058823529$; while $17 \times 0588\cdots = 9999999999999999$. So also with the repetend of $\frac{1}{29}$, which is 0344827586206896551724137931.

It is easy to see why, in reducing $1/p$ (p being a prime) to a decimal, the figures must begin to repeat in less than p decimal places; for at every step in the process of division the remainder must be less than the divisor. There are therefore only $p-1$ different numbers that can be remainder. After that the process repeats.

$$\frac{1}{7} = .1\frac{3}{7} = .14\frac{2}{7} = .142\frac{6}{7} = .1428\frac{4}{7} = .14285\frac{5}{7}$$
$$= .142857\frac{1}{7} = \ldots.$$

Hence if we multiply 142857 by 3, 2, 6, 4, 5, we get the repetend beginning after the 1st, 2d, 3d, 4th, 5th figures respectively.

"If a repetend contains $\dfrac{p-1}{2}$ digits, all the multiples up to $p-1$ will give one of two numbers each consisting of $\dfrac{p-1}{2}$ digits. Example: $\dfrac{1}{13} = .\dot{0}7692\dot{3}$.

$1 \times 76923 =$	76923	$2 \times 76923 =$	153846
$3 \times$ "	= 230769	$5 \times$ "	= 384615
$4 \times$ "	= 307692	$6 \times$ "	= 461538
$9 \times$ "	= 692307	$7 \times$ "	= 538461
$10 \times$ "	= 769230	$8 \times$ "	= 615384

$$12 \times \quad \text{``} \quad = 923076 \quad 11 \times \quad \text{``} \quad = 846153.\text{''}$$

(Escott.)

"In the repetend for $\frac{1}{7}$, if we divide the number into halves, their sum is composed of 9's, viz., $142 + 857 = 999$. A similar property is true of the repetend for $\frac{1}{17}$, etc. This property is true also of the two numbers obtained from $\frac{1}{13}$. However, when we find the repetends of fractions $1/p$ where the repetend contains only $\dfrac{p-1}{2}$ digits, but which is of the form $4n + 3$, it is not the halves of the numbers which are complementary, but the two numbers themselves. Example:

$$\frac{1}{31} = .\dot{0}3225806451612\dot{9} \qquad \frac{3}{31} = .\dot{0}9677419354838\dot{7}$$

$$\frac{30}{31} = .\dot{9}6774193548387\dot{0}$$

$$\text{Sum} = .\dot{9}9999999999999\dot{9}\text{''}$$

(Escott.)

"A useful application may be made of this property of repeating, in reducing a fraction $1/p$ to a decimal. After a number of figures have been found, as many more may be found by multiplying those already found by the remainder. It is, of course, advantageous to carry on the work until a comparatively small remainder has been found. Example: In reducing $\frac{1}{97}$ to a decimal, after we have obtained the digits .01030927835 we get a remainder 5. Therefore, from this point on the digits are the same as those of $\frac{1}{97}$ multiplied by 5. Multiplying by 5 (or dividing by 2) we get 11 more digits at once. The lengths of the periods of the reciprocals of primes have been determined at least as far as $p = 100{,}000$." (Escott.)

MULTIPLICATION AT SIGHT: A NEW TRICK WITH AN OLD PRINCIPLE.

This property of repeating the figures, possessed by these numbers, enables one to perform certain operations that seem marvelous till the observer understands the process. For example, one says: "I will write the multiplicand, you may write below it any multiplier you choose of—say—two or three figures, and I will immediately set down the complete product, writing from left to right." He writes for the multiplicand 142857. Suppose the observers then write 493 as the multiplier. He thinks of 493 × the number as $493/7 = 70\frac{3}{7}$; so he *writes* 70 as the first figures of the product (writing from left to right). Now 3/7 (i. e., $3 \times \frac{1}{7}$) is thought of as 3 × the repetend, and it is necessary to determine first where to begin in writing the figures in the circular order. This may be determined by thinking that, since 3 × 7(the units figure of the multiplicand) = 21, the last figure is 1; therefore the first figure is the next after 1 in the circular order, namely 4. (Or one may obtain the 4 by dividing 3 by 7.) So he *writes* in the product (after the 70) 4285. From the 71 remaining, the 70 first written must be subtracted (compare the explanation of 89 × 142857 given above). This leaves the last two figures 01, and the product stands 70428501. When the spectators have satisfied themselves by actual multiplication that this is the correct product, let us suppose that they test the "lightning calculator" with 825 as a multiplier. $825/7 = 117\frac{6}{7}$. *Write* 117. 6 × 7 = 42. Next figure after 2 in repetend is 8. *Write* 857. From the remaining 142 subtract the 117. *Write* 025.

Note that after the figures obtained by division (117 in the last example) have been written, there remain just six figures to write, and that the number first written is to be subtracted from the six-place number found from the circular order (117 subtracted from 857142 in the last example). After a little practice, products may be written in this way without hesitation.

If the multiplier is a multiple of 7, the process is even simpler.

Take 378 for multiplier. $378/7 = 54$. Think of it as $53\frac{7}{7}$. *Write* 53. $7 \times$ the repetend gives six nines. Mentally subtracting 53 from 999999, *write*, after the 53, 999946.

This trick may be varied in many ways, so as not to repeat. (Few such performances will bear repetition.) E. g., the operator may say, "I will give a multiplicand, you may write the multiplier, divide your product by 13, and I will write the quotient as soon as you have written the multiplier." He then writes as multiplicand 1857141, which is 13×142857 and is written instantly by the rule above. Now, as the 13 cancels, the quotient may be written as the product was written in the foregoing illustrations. Of course another number could have been used instead of 13.

A REPEATING TABLE.

Some peculiarities depending on the decimal notation of number. The first is the sum of the digits in the 9's table.

$$
\begin{aligned}
9 \times 1 &= 9 \\
9 \times 2 &= 18; \quad 1+8=9 \\
9 \times 3 &= 27; \quad 2+7=9 \\
9 \times 4 &= 36; \quad 3+6=9 \\
&\dots\dots\dots\dots \\
9 \times 9 &= 81; \quad 8+1=9 \\
9 \times 10 &= 90; \quad 9+0=9 \\
9 \times 11 &= 99; \quad 9+9=18; \quad 1+8=9 \\
9 \times 12 &= 108; \quad 1+0+8=9 \\
9 \times 13 &= 117; \quad 1+1+7=9 \\
&\text{etc.}
\end{aligned}
$$

The following are given by Lucas* in a note entitled *Multiplications curieuses*:

$$
\begin{aligned}
1 \times 9 + 2 &= 11 \\
12 \times 9 + 3 &= 111 \\
123 \times 9 + 4 &= 1111 \\
1234 \times 9 + 5 &= 11111 \\
12345 \times 9 + 6 &= 111111 \\
123456 \times 9 + 7 &= 1111111 \\
1234567 \times 9 + 8 &= 11111111 \\
12345678 \times 9 + 9 &= 111111111.
\end{aligned}
$$

* *Récréations Mathématiques*, IV, 232–3; *Théorie des Nombres*, I, 8.

$$9 \times 9 + 7 = 88$$
$$98 \times 9 + 6 = 888$$
$$987 \times 9 + 5 = 8888$$
$$9876 \times 9 + 4 = 88888$$
$$98765 \times 9 + 3 = 888888$$
$$987654 \times 9 + 2 = 8888888$$
$$9876543 \times 9 + 1 = 88888888$$
$$98765432 \times 9 + 0 = 888888888.$$
$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321.$$
$$12345679 \times 8 = 98765432$$
$$12345679 \times 9 = 111111111$$

to which may, of course, be added

$$12345679 \times 18 = 222222222$$
$$12345679 \times 27 = 333333333$$
$$12345679 \times 36 = 444444444$$
$$\text{etc.}$$

A FEW NUMERICAL CURIOSITIES.*

$$11^2 = 121; \quad 111^2 = 12321; \quad 1111^2 = 1234321; \quad \text{etc.}$$
$$1+2+1 = 2^2; \quad 1+2+3+2+1 = 3^2;$$
$$1+2+3+4+3+2+1 = 4^2; \quad \text{etc.}$$
$$121 = \frac{22 \times 22}{1+2+1}; \quad 12321 = \frac{333 \times 333}{1+2+3+2+1}; \quad \text{etc.}^\dagger$$

The following three consecutive numbers are probably the lowest that are divisible by cubes other than 1:

$$1375, \quad 1376, \quad 1377$$

(divisible by the cubes of 5, 2 and 3 respectively).

A curious property of 37 *and* 41. Certain multiples of 37 are still multiples of 37 when their figures are permuted cyclically: $259 = 7 \times 37$; $592 = 16 \times 37$; $925 = 25 \times 37$. The same is true of 185, 518, and 851; 296, 629, and 962. A similar property is true of multiples of 41: $17589 = 41 \times 429$; $75891 = 41 \times 1851$; $58917 = 41 \times 1437$; $89175 = 41 \times 2175$; $91758 = 41 \times 2238$.

Numbers differing from their logarithms only in the position of the decimal point. The determination of such numbers has been discussed by Euler and by Professor Tait. Following are three examples of a list that could be extended indefinitely.

$$\log 1.3712885742 = .13712885742$$
$$\log 237.5812087593 = 2.375812087593$$
$$\log 3550.2601815865 = 3.5502601815865.$$

*Nearly all of the numerical curiosities in this section were given to the writer by Mr. Escott.

† *The Monist*, 1906; XVI, 625.

Powers having same digits.

Consecutive numbers whose squares have the same digits:

$$13^2 = 169 \qquad 157^2 = 24649 \qquad 913^2 = 833569$$
$$14^2 = 196 \qquad 158^2 = 24964 \qquad 914^2 = 835396.$$

Cubes containing the same digits:

$$345^3 = 41063625 \qquad 331^3 = 36264691$$
$$384^3 = 56623104 \qquad 406^3 = 66923416$$
$$405^3 = 66430125.$$

A pair of numbers two of whose powers are composed of the same digits:

$$32^2 = 1024 \qquad 32^4 = 1048576$$
$$49^2 = 2401 \qquad 49^4 = 5764801$$

Square numbers containing the digits not repeated.

1. Containing the nine digits:*

$$11826^2 = 139854276 \qquad 20316^2 = 412739856$$
$$12363^2 = 152843769 \qquad 22887^2 = 523814769$$
$$12543^2 = 157326849 \qquad 23019^2 = 529874361$$
$$14676^2 = 215384976 \qquad 23178^2 = 537219684$$
$$15681^2 = 245893761 \qquad 23439^2 = 549386721$$
$$15963^2 = 254817369 \qquad 24237^2 = 587432169$$
$$18072^2 = 326597184 \qquad 24276^2 = 589324176$$
$$19023^2 = 361874529 \qquad 24441^2 = 597362481$$
$$19377^2 = 375468129 \qquad 24807^2 = 615387249$$

*Published in the *Mathematical Magazine*, Washington, D.C., in 1883, and completed in *L'Intermédiaire des Mathématiciens*, 1897 (4:168).

$19569^2 = 382945761$	$25059^2 = 627953481$
$19629^2 = 385297641$	$25572^2 = 653927184$
$25941^2 = 672935481$	$27273^2 = 743816529$
$26409^2 = 697435281$	$29034^2 = 842973156$
$26733^2 = 714653289$	$29106^2 = 847159236$
$27129^2 = 735982641$	$30384^2 = 923187456$

2. Containing the ten digits:*

$32043^2 = 1026753849$	$45624^2 = 2081549376$
$32286^2 = 1042385796$	$55446^2 = 3074258916$
$33144^2 = 1098524736$	$68763^2 = 4728350169$
$35172^2 = 1237069584$	$83919^2 = 7042398561$
$39147^2 = 1532487609$	$99066^2 = 9814072356$

Arrangements of the digits.

If the number 123456789 be multiplied by all the integers less than 9 and prime to 9, namely 2, 4, 5, 7, 8, each product contains the nine digits and uses each digit but once.

Each term in the following subtraction contains each of the nine digits once.

$$\begin{array}{r} 987654321 \\ 123456789 \\ \hline 864197532 \end{array}$$

* *L'Intermédiaire des Mathématiciens*, 1907 (14:135).

To arrange the nine digits additively so as to make 100:

$$\begin{array}{r} 15 \\ 36 \\ 47 \\ \hline 98 \\ 2 \\ \hline 100 \end{array} \qquad \begin{array}{r} 56 \\ 8 \\ 4 \\ 3 \\ \hline 71 \\ 29 \\ \hline 100 \end{array} \qquad \begin{array}{r} 95\frac{1}{2} \\ 4\frac{38}{76} \\ \hline 100 \end{array}$$

Many other solutions. See Fourrey and Lucas.

To arrange the ten digits additively so as to make 100:

$$\begin{array}{r} 50\frac{1}{2} \\ 49\frac{38}{76} \\ \hline 100 \end{array} \qquad \begin{array}{r} 80\frac{27}{54} \\ 19\frac{3}{6} \\ \hline 100 \end{array}$$

Many ways of doing this also.

To place the ten digits so as to produce each of the digits:

$$\frac{62}{31} - \frac{970}{485} = 0 \qquad \frac{13485}{02697} = 5$$
$$\frac{62}{31} \times \frac{485}{970} = 1 \qquad \frac{34182}{05697} = 6$$
$$\frac{97062}{48531} = 2 \qquad \frac{41832}{05976} = 7$$
$$\frac{107469}{35823} = 3 \qquad \frac{25496}{03187} = 8$$
$$\frac{23184}{05796} = 4 \qquad \frac{57429}{06381} = 9 = \frac{95742}{10638}.$$

Lucas* also gives examples where the ten digits are used, the zero not occupying the first place in a number, for all of the ten numbers above except 6, which is impossible. It will be noticed that, in the example given above for 3, the digit 3 occurs twice.

* *Théorie des Nombres*, p. 40.

The nine digits arranged to form a perfect cube:

$$\frac{8}{32461759} = \left(\frac{2}{319}\right)^3 \qquad \frac{8}{24137569} = \left(\frac{2}{289}\right)^3$$
$$\frac{125}{438976} = \left(\frac{5}{76}\right)^3 \qquad \frac{512}{438976} = \left(\frac{8}{76}\right)^3.$$

The ten digits arranged to form a perfect cube:

$$\frac{9261}{804357} = \left(\frac{21}{93}\right)^3.$$

The ten digits placed to give an approximate value of π:

$$\pi = \frac{67389}{21450} = 3.141678 + .$$

*Fourier's method of division** by a number of two digits of which the units digit is 9. Increase the divisor by 1, and increase the dividend used at each step of the operation by the quotient figure for that step. E. g., $43268 \div 29$. The ordinary arrangement is shown at the left

$$\begin{array}{r} 1492 \\ 29\overline{)43268} \\ 29 \\ \hline 142 \\ 116 \\ \hline 266 \\ 261 \\ \hline 58 \\ 58 \\ \hline \end{array} \qquad \begin{array}{r} 29)43268 \\ \hline 1492 \end{array}$$

*Fourier, p. 187.

for comparison. The form at the right is all that need be written in Fourier's method. To perform the operation, one thinks of the divisor as 30; $4 \div 3$, $(43 \div 30,)$ 1; write the 1 in the quotient and add it to the 43; $44 - 30 = 14$; $14 \div 3$, 4; etc. The reason underlying it is easily seen. E. g., at the second step we have, by the common method, $142 - 4 \times 29$. By Fourier's method we have $142 + 4 - 4 \times 30$. The addition of the same number (the quotient figure) to both minuend and subtrahend does not affect the remainder.

In the customary method for the foregoing example one practically uses 30 as divisor in determining the quotient figure (thinking at the second step, $14 \div 3$, 4). In Fourier's method this is extended to the whole operation and the work is reduced to mere short division.

So also in dividing by 19, 39, 49, etc. The method is, of course, not limited to divisors of two places, nor to those ending in 9. It may be used in dividing by a number ending in 8, 7, etc. by increasing the divisor by 2, 3, etc., and also the dividend used at each step by 2, 3, etc. times the quotient figure for that step. But the advantage of the method lies chiefly in the case first stated.

"The method is rediscovered every little while by some one and hailed as a great discovery."

NINE.

Curious properties of the number nine, and numerical tricks with it, are given and explained by many writers; among them Dr. Edward Brooks, in his *Philosophy of Arithmetic*. Of all such properties, perhaps the most practical application is the check on division and multiplication by casting out nines, the Hindu check as it is called. Next might come the bookkeeper's clue to inverted numbers. In double-entry bookkeeping, if there has been inversion (e. g., $4.83 written in the debit side of one account, and $4.38 in the credit side of another) and no other mistake, the trial balance will be "off" by a multiple of nine. It can also be seen in what columns the transposition was made.

Recently suggested, and of no practical interest, is another property of the "magic number," easily explained, like the rest, but at first glance curious: invert the figures of any three-place number; divide the difference between the original number and the inverted number by nine; and you may read the quotient forward or backward. Moreover the figure that occurs in the quotient is the difference between the first and last figures of the number taken.

$$\begin{array}{r} 845 \\ 548 \\ \hline 9\overline{)297} \\ \hline 33 \end{array}$$

Explanation: Let a, b, c be the hundreds, tens, units figures respectively of any three-place number. Then the number is $100a + 10b + c$, and the number inverted is $100c + 10b + a$.

$$\begin{aligned} \frac{(100a + 10b + c) - (100c + 10b + a)}{9} &= \frac{99(a - c)}{9} \\ &= 11(a - c). \end{aligned}$$

The product of 11 and any one-place number will have both figures alike, and may be read either way.

Better known are the following three—all old and all depending on the principle, that the remainder, after dividing any number by 9, is the same as the remainder after dividing the sum of its digits by 9.

1. Find the difference between a number of two figures and the number made by inverting the figures, conceal the numbers from me, but tell me one figure of the difference. I will tell you whether there is another figure in the difference, and, if so, what it is. (This can scarcely be repeated without every spectator noticing that one merely subtracts the given figure of the difference from 9.)

2. Write a number of three or more places, divide by 9, and tell me the remainder; erase one figure, not zero, divide the resulting number by 9, and tell me the remainder. I will tell you the figure erased (which is, of course, the first remainder minus the second, or if the first is not greater than the second, then the first $+9 -$ the second).

3. Write a number with a missing figure, and I will immediately fill in the figure necessary to make the number exactly divisible by 9. (Suppose 728 57 to be written. Write 7 in the space; for the excess from the given number after casting out 9's is 2, and $9 - 2 = 7$.) This may be varied by undertaking to fill the space with a figure that shall make the number divisible by nine and leaving a specified remainder.*

*Adapted from Hooper, I, 22.

FAMILIAR TRICKS BASED ON LITERAL ARITHMETIC.

Besides the tricks with the number 9, there are many other well-known arithmetical diversions, most, but not all, of them, depending on the Arabic notation of numbers used. Those illustrated in this section are specially numerous, can be "made while you wait" by any one with a little ingenuity and an elementary knowledge of algebra (or, more properly, of literal arithmetic) and, when set forth, are transparent the moment they are expressed in literal notation. They are amusing to children, and it is no wonder that the supply of them is perennial. The following three may be given as fairly good types. The first two are taken from Dr. Hooper's book, which was published in 1774. Verbatim quotation of them is made in order to preserve the flavor of quaintness. Only the explanation in terms of literal arithmetic is by the present writer.

1. *A person privately fixing on any number, to tell him that number.*

After the person has fixed on a number, bid him double it and add 4 to that sum, then multiply the whole by 5; to the product let him add 12, and multiply the amount by 10. From the sum of the whole let him deduct 320, and tell you the remainder, from which, if you cut off the two last figures, the number that remains will be that he fixed on.

Let n represent any number selected. The first member of the following equality readily reduces to n, and the identity explains the trick.

$$\big\{[(2n+4)5+12]10-320\big\}\div 100=n.$$

2. *Three dice being thrown on a table, to tell the number of each of them, and the order in which they stand.*

Let the person who has thrown the dice double the number of that next his left hand, and add 5 to that sum; then multiply the amount by 5, and to the product add the number of the middle die; then let the whole be multiplied by 10, and to that product add the number of the third die. From the total let there be subtracted 250, and the figures of the number that remains will answer to the points of the three dice as they stand on the table.

Let x, y, z represent the numbers of points shown on the three dice in order. Then the instructions, expressed in symbols, give

$$[(2x + 5)5 + y]10 + z - 250.$$

Removing signs of grouping, we have

$$100x + 10y + z,$$

the number represented by the three digits x, y, z in order.

3. "Take the number of the month in which you were born (1 for January, 2 for February, etc.), double it; add 5; multiply by 50; add your age in years; subtract 365; add 115. The resulting number indicates your age—month and years." E. g., a person 19 years old and born in August (8th month) would have, at the successive stages of the operation, 8, 16, 21, 1050, 1069, 704; and for the final number, 819 (8 for August, 19 for the years).

If we let m represent the number of the month, and y the number of years, we can express the rule as a formula:

$$(2m + 5)50 + y - 365 + 115,$$

which simplifies to

$$100m + y,$$

the number of hundreds being the number of the month, and the number expressed by the last two digits being the number of years.

GENERAL TEST OF DIVISIBILITY.*

Let M represent any integer containing no prime factor that is not a factor of 10 (that is, no primes but 5 and 2). Then 1/M expressed decimally is terminate. Call the number of places in the decimal m. Let N represent any prime except 5, 2, 1. Then the reciprocal of N expressed decimally is a circulate. Call the number of places in the repetend n.

1. The remainder obtained by dividing any integer, I, by M is the same as that obtained by dividing the number represented by the last (right-hand) m digits of I by M. If the number represented by those m digits is divisible by M, I is divisible by M, and not otherwise.

2. The remainder obtained by dividing I by N is the same as that obtained by dividing the sum of the numbers expressed by the successive periods of n digits of I by N. If that sum is divisible by N, I is divisible by N, and not otherwise. This depends on Fermat's theorem, that $P^{p-1} - 1$ is divisible by p when p and P are prime to each other.

3. If a number is composite and contains a prime factor other than 5 and 2, the divisibility of I by it may be tested by testing with the factors separately by (1) and (2).

Thus it is possible to test the divisibility of any integer by any other integer. This is usually of only theoretic interest, as actual division is preferable. But in the case of 2, 3, 4, 5, 6, 8, 9, and 10 the test is easy and practical. A simple statement of it for each of these particular cases is found in almost any arithmetic.

For 11 a test slightly easier than the special application of the general test is usually given. That is, subtract the sum of the even-numbered digits from the sum of the odd-numbered digits (counting from the right) and add 11 to the minuend if smaller than the subtrahend. The result gives the same remainder when divided by 11 as the original number gives. The original number is divisible by 11 if that result is, and not otherwise. These remainders may be used in the same

*Divisible *without remainder* is of course the meaning of "divisible" in such a connection.

manner as the remainders used in casting out the nines, but are not so conveniently obtained.

Test of divisibility by 7. No known form of the general test in this case is as easy as actually dividing by 7. From the point of view of theory it may be worth noticing that, as 7's reciprocal gives a complementary repetend, the general test admits of variety of form.* Let us consider, however, the direct application.

Since the repetend has 6 places, the test for divisibility by 7 is as follows: A number is divisible by 7 if the sum of the numbers represented by the successive periods of 6 figures each is divisible by 7, and not otherwise; e. g.,

Given the number 26,436,080,216,581.

$$\begin{array}{r@{}r} & 216581 \\ & 436080 \\ & 26 \\ \hline 7) & 652687 \\ \hline & 93241. \end{array}$$

No remainder; therefore the given number is divisible by 7.

Test of divisibility by 7, 11, and 13 *at the same time.*† Since $7 \times 11 \times 13 = 1001$, divide the given number by 1001. If the remainder is divisible by 7, 11, or 13, the given number is also, and not otherwise.

To divide by 1001, subtract each digit from the third digit following. An inspection of a division by 1001 will show why this simple rule holds. The method may be made clear by an example, 4,728,350,169 ÷ 1001.

$$\begin{array}{l} 4728350169 \\ 472\not{4}626543 \\ 3 \end{array}$$

Quotient, 4723626; remainder, 543.

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*A chapter of Brooks's *Philosophy of Arithmetic* is devoted to divisibility by 7.

†This was given to the author by Mr. Escott, who writes: "I have never seen it published, but it is so simple that it would be surprising if it had not been."

The third digit before the 4 being 0 (understood), write the difference, 4, beneath the 4. Similarly for 7 and 2. $8-4=4$ (which for illustration is here written beneath the 8). We should next have $3-7$. This changes the 4 just found to 3, and puts 6 under the original 3 (that is, $83-47=36$). $5-2$, $0-3$ (always subtracting from a digit of the *original* number the third digit to the left in the *difference*, or lower, number), $1-6$, etc. Making the corrections mentally we have the number as written. The number represented by the last three digits, 543, is the *remainder* after dividing the given number by 1001, and the number represented by the other digits, 4723626, is the *quotient*. With a little practice, this method can be applied rapidly and without making erasures. The remainder, 543, which alone is needed for the test, may also be obtained by subtracting the sum of the even-numbered periods of three figures each in the original number from the sum of the odd-numbered periods. A rapid method of obtaining the remainder thus is easily acquired; but the way illustrated above is more convenient.

However obtained, the remainder is divisible or not by 7, 11, or 13, according as the given number is divisible or not. (Here 543 is not divisible by 7, 11, or 13; therefore 4728350169 is not divisible by either of them.) The original number is thus replaced, for the purpose of investigation, by a number of three places at most. As this tests for three common primes at once, it is convenient for one factoring large numbers without a factor table.

MISCELLANEOUS NOTES ON NUMBER.

The theory of numbers has been called a "neglected but singularly fascinating subject."[*] "Magic charm" is the quality ascribed to it by the foremost mathematician of the nineteenth century.[†] Gauss said also: "Mathematics the queen of the sciences, and arithmetic [i. e., theory of numbers] the crown of mathematics." And he was master of the sciences of his time. "While it requires some facility in abstract reasoning, it may be taken up with practically no technical mathematics, is easily amenable to numerical exemplifications, and leads readily to the frontier. It is perhaps the only branch of mathematics where there is any possibility that new and valuable discoveries might be made without an extensive acquaintance with technical mathematics."[‡]

An interesting exercise in higher arithmetic is to investigate theorems and the established properties of particular numbers to determine which have their origin in the nature of number itself and which are due to the decimal scale in which the numbers are expressed.

Fermat's last theorem. Of the many theorems in numbers discovered by Fermat, nearly all have since been proved. A well-known exception is sometimes called his "last theorem." It "is to the effect that no integral values of x, y, z can be found to satisfy the equation $x^n + y^n = z^n$, if n is an integer greater than 2. This proposition has acquired extraordinary celebrity from the fact that no general demonstration of it has been given, but there is no reason to doubt that it is true."[§] It has been

[*] Ball, *Hist.*, p. 416.

[†] "The most beautiful theorems of higher arithmetic have this peculiarity, that they are easily discovered by induction, while on the other hand their demonstrations lie in exceeding obscurity and can be ferreted out only by very searching investigations. It is precisely this which gives to higher arithmetic that magic charm which has made it the favorite science of leading mathematicians, not to mention its inexhaustible richness, wherein it so far excels all other parts of pure mathematics." (Gauss; quoted by Young, p. 155.)

[‡] Young, p. 155.

[§] Ball, *Recreations*, p. 37.

proved for special cases, and proved generally if certain assumptions be granted. Fermat asserted that he had a valid proof. That may yet be rediscovered; or, more likely, a new proof will be found by some new method of attack. "Interest in problems connected with the theory of numbers seems recently to have flagged, and possibly it may be found hereafter that the subject is approached better on other lines."*

Wilson's theorem may be stated as follows: If p is a prime, $1+(p-1)!$ is a multiple of p. This well-known proposition was enunciated by Wilson,† first published by Waring in his *Meditationes Algebraicæ*, and first proved by Lagrange in 1771.

Formulas for prime numbers. "It is easily demonstrated that no rational algebraic formula can always, give primes. Several remarkable expressions have been found, however, which give a large number of primes for consecutive values of x. Legendre gave $2x^2+29$, which gives primes for $x=0$ to 28, or for 29 values of x. Euler found x^2+x+41, which gives primes for $x=0$ to 39, i. e., 40 values of x. I have found $6x^2+6x+31$, giving primes for 29 values of x; and $3x^2+3x+23$, giving primes for 22 values of x. These expressions give different primes. We can transform them so that they will give primes for more values of x, but not different primes. For instance, in Euler's formula if we replace x by $x-40$, we get $x^2-79x+1601$, which gives primes for 80 consecutive values of x." (Escott.)

A Chinese criterion for prime numbers. With reference to the so-called criterion, that a number p is prime when the condition, that $2^{p-1}-1$ be divisible by p, is satisfied, Mr. Escott makes the following

*Ball, *Hist.*, p. 469.

†As he was not a professional mathematician, but little mention of him is made in histories of the subject. The following items may be of interest. They are from De Morgan's *Budget of Paradoxes*, p. 132–3. John Wilson (1741–1793) was educated at Cambridge. While an undergraduate he "was considered stronger in algebra than any one in the University, except Professor Waring, one of the most powerful algebraists of the century." Wilson was the senior wrangler of 1761. He entered the law, became a judge, and attained a high reputation.

interesting comment:

"This is a well-known property of prime numbers (Fermat's Theorem) but it is not sufficient. My attention was drawn to the problem by a question in *L'Intermédiaire des Mathématiciens*, which led to a little article by me in the *Messenger of Mathematics.* As the smallest number which satisfies the condition and which is not prime is 341, and to verify it by ordinary arithmetic (not having the resources of the Theory of Numbers) would require the division of $2^{340} - 1$ by 341, it is probable that the Chinese obtained the test by a mere induction."

Are there more than one set of prime factors of a number? Most text-books answer no; and this answer is correct if only arithmetic numbers are considered. But when the conception of number is extended to include complex numbers, the proposition, that a number can be factored into prime factors in only one way, ceases to hold. E. g., $26 = 2 \times 13 = (5 + \sqrt{-1})(5 - \sqrt{-1})$.

Asymptotic laws. This happily chosen name describes "laws which approximate more closely to accuracy as the numbers concerned become larger."* Legendre is among the best-known names here. One of the most celebrated of the original researches of Dirichlet, in the middle of the last century, was on this branch of the theory of numbers.

Growth of the concept of number, from the arithmetic integers of the Greeks, through the rational fractions of Diophantus, ratios and irrationals recognized as numbers in the sixteenth century, negative *versus* positive numbers fully grasped by Girard and Descartes, imaginary and complex by Argand, Wessel, Euler and Gauss,† has proceeded in recent times to new theories of irrationals and the establishing of the continuity of numbers without borrowing it from space.‡

Some results of permutation problems. The formulas for the number of permutations, and the number of combinations, of n dissimilar things

*Ball, *Hist.*, p. 464.

†See p. 76.

‡See Cajori's admirable summary, *Hist. of Math.*, p. 372.

taken r at a time are given in every higher algebra. The most important may be condensed into one equality:

$$^n\mathrm{P}_r = n(n-1)(n-2)\dots(n-r+1) = \frac{n!}{(n-r)!} = {}^n\mathrm{C}_r r!.$$

There are 3,979,614,965,760 ways of arranging a set of 28 dominoes (i. e., a set from double zero to double six) in a line, with like numbers in contact.*

"Suppose the letters of the alphabet to be wrote so small that no one of them shall take up more space than the hundredth part of a square inch: to find how many square yards it would require to write all the permutations of the 24 letters in that size."† Dr. Hooper computes that "it would require a surface 18620 times as large as that of the earth to write all the permutations of the 24 letters in the size above mentioned."

Fear has been expressed that if the epidemic of organizing societies should persist, the combinations and permutations of initial letters might become exhausted. We have F.A.A.M., I.O.O.F., K.M.B., K.P., I.O.G.T., W.C.T.U., Y.M.C.A., Y.W.C.A., A.B.A., A.B.S., A.C.M.S., etc., etc. An almanac names more than a hundred as "prominent in New York City," and its list is exclusive of fraternal organizations, of which the number is known to be vast. Already there are cases of two societies having names with the same initial letters. But by judicious choice this can long be avoided. Hooper's calculation supposed the entire alphabet to be employed in every combination. Societies usually employ only 2, 3 or 4 letters. And a letter may repeat, as the *A* in the title of the A.L.A. or of the A.A.A. The present problem is therefore different from that above. The number of permutations of 26 letters taken two at a time, the two being not necessarily dissimilar, is 26^2; three at a time, 26^3; etc. As there is occasionally a society known by one letter and occasionally one known by five, we have

$$26^1 + 26^2 + 26^3 + 26^4 + 26^5 = 12{,}356{,}630.$$

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*Ball, *Recreations*, p. 30, citing Reiss, *Annali di matematica*, Milan, Nov. 1871.
†Hooper, I, 59.

This total of possible permutations is easily beyond immediate needs. By lengthening the names of societies (as seems to have already begun) the total can be made much larger. Since the time when Hooper's calculations were made, two letters have been added to the alphabet. When the number of societies reaches about the twelve million mark, it would be well to agitate for a further extension of the alphabet. With these possibilities one may be assured, on the authority of exact science, that there is no cause for immediate alarm. The author hastens to allay the apprehensions of prospective organizers.

Tables. Many computations would not be possible without the aid of tables. Some of them are monuments to the patient application of their makers. Once made, they are a permanent possession. The time saved to the computer who uses the table is the one item taken into account in judging of the value of a table. It is difficult to appreciate the variety and extent of the work that has been done in constructing tables. For this purpose an examination of Professor Glaisher's article "Tables" in the *Encyclopædia Britannica* is instructive.

Anything that facilitates the use of a book of tables is important. Spacing, marginal tabs (in-cuts), projecting tabs—all such devices economize a little time at each handling of the book; and in the aggregate this economy is no trifle. Among American collections of tables for use in elementary mathematics the best example of convenience of arrangement for ready reference is doubtless Taylor's *Five-place Logarithmic and Trigonometric Tables* (1905). Dietrichkeit's *Siebenstellige Logarithmen und Antilogarithmen* (1903) is a model of convenience.

When logarithms to many places are needed, they can be readily calculated by means of tables made for the purpose, such as Gray's for carrying them to 24 places (London, 1876).

Factor tables have been printed which enable one to resolve into prime factors any composite number as far as the 10th million. They were computed by different calculators. "Prof. D. N. Lehmer, of the University of California, is now at work on factor tables which will extend to the 12th million. When completed they will be published by the Carnegie Institution, Washington, D.C. According to Petzval, tables giving the smallest prime factors of numbers as far as 100,000,000 have been calculated by Kulik, but have remained in manuscript in the possession of the Vienna Academy. . . Lebesgue's *Table des Diviseurs des Nombres* goes as far as 115500 and is very compact, occupying only 20 pages." (Escott.)

Some long numbers. The computation of the value of π to 707 decimal places by Shanks* and of e to 346 places by Boorman,† are famous feats of calculation.

"Paradoxes of calculation sometimes appear as illustrations of the value of a new method. In 1863, Mr. G. Suffield, M.A., and Mr. J. R. Lunn, M.A., of Clare College and of St. John's College, Cambridge, published the whole quotient of 10000 . . . divided by 7699, throughout the whole of one of the recurring periods, having 7698 digits. This was done in illustration of Mr. Suffield's method of synthetic division."‡

Exceptions have been found to Fermat's theorem on binary powers (which he was careful to say he had not proved). The theorem is, that all numbers of the form $2^{2^n} + 1$ are prime. Euler showed, in 1732, that if $n = 5$, the formula gives 4,294,967,297, which $= 641 \times$ 6,700,417. "During the last thirty years it has been shown that the resulting numbers are composite when $n = 6, 9, 11, 12, 18, 23, 36,$ and 38; the two last numbers contain many thousands of millions of digits."§ To these values of n for which $2^{2^n} + 1$ is composite, must now be added the value $n = 73$. "Dr. J. C. Morehead has proved this year [1907] that this number is divisible by the prime number $2^{75} \cdot 5 + 1$. This last number contains 24 digits and is probably the largest prime

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*See page 105.

†Mathematical Magazine, 1:204.

‡De Morgan, p. 292. "Suffield's 'new' method was discovered by Fourier in the early part of the century and has been rediscovered many times since. It was published, apparently as a new discovery, a few years ago in the *Mathematical Gazette*." (Escott.)

§Ball, *Recreations*, p. 37.

number discovered up to the present."* If the number $2^{2^{73}} + 1$ itself were written in the ordinary notation without exponents, and if it were desired to print the number in figures the size of those on this page, how many volumes like this would be required? They would make a library many millions of times as large as the Library of Congress.

How may a particular number arise? (1) From purely mathematical analysis—in the investigation of the properties of numbers, as in the illustrations just given, in the investigation of the properties of some ideally constructed magnitude, as the ratio of the diagonal to the side of a square, or in any investigation involving only mathematical elements; (2) from measurement of actual magnitude, time etc.: (3) by arbitrary invention, as when a text-book writer or a teacher makes examples; or (4) by combinations of these.

Those of class (3) are generally used to develop skill in the manipulation of numbers from classes (1) and (2).

Numbers from source (2), measurement, are the subject of the next section.

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*Mr. Escott.

NUMBERS ARISING FROM MEASUREMENT.

There is no such thing as an exact measurement of distance, capacity, mass, time, or any such quantity. It is only a question of *degree* of accuracy.

"The best time-pieces can be trusted to measure a week within one part in 756,000."* The equations of standards on page 132 show the degree of accuracy attained in two instances by the International Bureau of Weights and Measures. In the measure of length (the distance between two lines on a bar of platinum-iridium) the range of error is shown to be 0.2 in a million, or one in five million. In the measure of mass it is one in five hundred million. But these are measurements famous for their precision, made in cases in which accuracy is of prime importance, and the comparisons effected under the most favorable conditions. No such accuracy is attained in most work. In a certain technical school, two-tenths of a per cent is held to be fair tolerance of error for "exact work" in chemical analysis. The accuracy in measurement attained by ordinary artisans in their work is of a somewhat lower degree.

Now in a number expressing measurement the number of significant figures indicates the degree of accuracy. Hence the number of significant digits is limited. If any one were to assert that the distance of Neptune from the sun is 2,788,820,653 miles, the statement would be immediately rejected. A distance of billions of miles can not, by any means now known, be measured to the mile. We should be sure that the last four or five figures must be unknown and that this number is not to be taken seriously. What astronomers do state is that the distance is 2,788,800,000 miles.

The metrology of the future will doubtless be able to extend gradually the limits of precision, and therefore to expand the significant parts of numbers. But the principle will always hold.

The numbers arising from the measurements of daily life have but few significant figures.

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*Prof. William Harkness, "Art of Weighing and Measuring," *Smithsonian Report* for 1888, p. 616.

The following paragraph is another illustration of the principle.

Decimals as indexes of degree of accuracy of measure. The child is taught that .42 = .420 = .4200. True; but the scientist who reports that a certain distance is .42 cm, and the scientist who reports it as .420 cm, wish to convey, and do convey, to their readers different impressions. From the first we understand that the distance is .42 cm correct to the nearest hundredth of a cm; that is, it is more than .415 cm and less than .425 cm. From the second we learn that it is .420 cm to the nearest thousandth; that is, more than .4195 and less than .4205. Compare the decimals, including 0.00100, in the equation of the U.S. standard meter, p. 132.

Exact measurement is an ideal. It is the *limit* which an ever improving metrology is approaching forever nearer. The question always is of *degree* of accuracy of measure. And this question is answered by the number of decimal places in which the result is expressed.

Some applications. The foregoing principle explains why for very large and very small numbers the index notation is sufficient; in which it is said, for example, that a certain star is $5x10^{13}$ miles from the earth. This is easier to write than 5 followed by 13 ciphers, and there is no need to enumerate and read such a number. Similarly 10 with a negative exponent serves to write such a decimal fraction as is used to express the length of a wave of light or any of the minute measurements of microscopy.

The principle explains also why a table of logarithms for ordinary use need not tabulate numbers beyond four or five places (four or five places in the "arguments," to use the technical term of table makers; only the logarithms of numbers to 10,000, or 100,000, to use the common phraseology). Interpolation extends them to one more place with fair accuracy, and for ordinary computation one rarely needs the logarithm of a number of more than five significant digits.

It explains also why a method of approximation in multiplication is so desirable. If any of the data are furnished by measurement, the result can be only approximate at best. Example VII on page 45, explained on page 44, is a case in point. To compute that product to six decimal places would waste time. Worse than that; to show such a result would *pretend* to an accuracy *not attained*, by conveying the impression that the circumference is known to six decimal places when in fact it is known to but two decimal places.*

In a certain village the tax rate, found by dividing the total appropriation for the year by the total assessed valuation, was .01981 for the year 1906. As always (unless the divisor contains no prime factor but 2 and 5) the quotient is an interminate decimal. To how many places should the decimal be carried? Theoretically it should be carried far enough to give a product "correct to cents" when used to compute the tax of the highest taxpayer. In this case the decimal is accurate enough for all assessments not exceeding $1000. As a matter of fact, there were several in excess of this amount.

For an understanding of the common applications of arithmetic it is important that the learner appreciate the elementary considerations of the theory of error; at least that he habitually ask himself, "To how many places may my result be regarded as accurate?"

*Even the second decimal place is in doubt, as may be seen by taking for multiplicand first 74.276, then 74.284.

The enormous results obtained by computing compound interest—as well as the wide divergence between these or any results obtained from a geometric progression of many terms and the results found in actual life—may be seen from the following "examples":

At 3% (the prevailing rate at present in savings banks) $1 put at interest at the beginning of the Christian era to be compounded annually would now amount to $\$(1.03)^{1906}$, which by the use of logarithms is found to be, in *round* numbers, $3,000,000,000,000,000,000,000,000. The amount of $1 for the same time and rate but at *simple* interest would be only $58.18.

*If the Indians hadn't spent the $*24. In 1626 Peter Minuit, first governor of New Netherland, purchased Manhattan Island from the Indians for about $24. The rate of interest on money is higher in new countries, and gradually decreases as wealth accumulates. Within the present generation the legal rate in the state has fallen from 7% to 6%. Assume for simplicity a uniform rate of 7% from 1626 to the present, and suppose that the Indians had put their $24 at interest at that rate (banking facilities in New York being always taken for granted!) and had added the interest to the principal yearly. What would be the amount now, after 280 years? 24×1.07^{280} = more than 4,042,000,000.

The latest tax assessment available at the time of writing gives the realty for the borough of Manhattan as $3,820,754,181. This is estimated to be 78% of the actual value, making the actual value a little more than $4,898,400,000.

The amount of the Indians' money would therefore be more than the present assessed valuation but less than the actual valuation. The Indians could have bought back most of the property now, with improvements; from which one might point the moral of saving money and putting it at interest! The rise in the value of the real estate of Manhattan, phenomenal as it is, has but little more than kept pace with the growth of money at 7% compound interest. But New York realty values are now growing more rapidly: the Indians would better purchase soon!

DECIMAL SEPARATRIXES.

The term *separatrix* in the sense of a mark between the integral and fractional parts of a number written decimally, was used by Oughtred in 1631. He used the mark ∟ for the purpose. Stevin had used a figure in a circle over or under each decimal place to indicate the order of that decimal place. Of the various other separatrixes that have been used, four are in common use to-day, if (2) and (3) below may be counted separately:

1. *A vertical line*: e. g., that separating cents from dollars in ledgers, bills, etc. As a temporary separatrix the line appears in a work by Richard Witt, 1613. Napier used the line in his *Rabdologia*, 1617. This is a very common separatrix in every civilized country to-day.

2. The *period*. Fink, citing Cantor, says that the decimal point is found in the trigonometric tables of Pitiscus (in Germany) 1612. Napier, in the *Rabdologia*, speaks of using the period or comma. His usage, however, is mostly of a notation now obsolete (but he uses the comma at least once). The period has always been the prevailing form of the decimal point in America.

3. The *Greek colon* (dot above the line). Newton advocated placing the point in this position "to prevent it from being confounded with the period used as a mark of punctuation" (Brooks). It is commonly so written in England now.

4. The *comma*. The first known instance of its use as decimal separatrix is said to be in the Italian trigonometry of Pitiscus, 1608. Perhaps next by Kepler, 1616, from which may be dated the German use of it. Briggs used it in his table of logarithms in 1624, and early English writers generally employed the comma. English usage changed to the Greek colon; but the comma is the customary form of the decimal point on the continent of Europe.

The usage as to decimal point is not absolutely uniform in any of the countries named; but, in general, one expects to see $1\frac{25}{100}$ written decimally in the form of 1.25 in America, $1 \cdot 25$ in England, and 1,25 in Germany, France or Italy.

A mere space to indicate the separation may also be mentioned as common in print.

The vertical line (for a column of decimals) and the space should doubtless persist, and *one* form of the "point." Prof. G. A. Miller, of the University of Illinois, who argues for the comma as being the symbol used by much the largest number of mathematicians, remarks:* "As mathematics is pre-eminently cosmopolitan and eternal it is very important that its symbols should be world symbols. All national distinctions along this line should be obliterated as rapidly as possible."

*"On Some of the Symbols of Elementary Mathematics," *School Science and Mathematics*, May, 1907.

Where the decimal point is a comma the separation of long numbers into periods of three (or six) figures for convenience of reading is effected by spacing.

PRESENT TRENDS IN ARITHMETIC.

"History is past politics, and politics is present history." Such is the apothegm of the famous historian Freeman. In the case of a science and an art, like arithmetic or the teaching of arithmetic, history is past method, and method is present history. The fact that our generation is helping to make the history of arithmetic and of the teaching of arithmetic—as it is also making history in other matters that attract more public attention—is the reason for considering now some of the present trends in arithmetic. A present trend is a pointer pointing from what has been to what is to be, since the science is a continuum. Lord Bolingbroke said that we study history to know how to act in the future, to make the most of the future. That is why we study history in the making, or present trends, in so far as it is possible for us, living in the midst, to see those trends.

Very noticeable among them is the gradual decimalization of arithmetic. Counting by 10 is prehistoric in nearly all parts of the world, 10 fingers being the evident explanation. If we had been present at the beginning of arithmetical history, we might have given the primitive race valuable advice as to the choice of a radix of notation! It would then have been opportune to call attention to the advantage of 12 over 10 arising from the greater factorability of 12. Or if the pioneers of arithmetic had been like the Gath giant of 2 Sam. 21:20, with six fingers on each hand, they would doubtless have used 12 as a radix. Lacking such counsel, and being equipped by nature with only 10 fingers to use as counters, they started arithmetic on a decimal basis. History since has been a steady progress in the direction thus chosen (except in details like the table of time, where the incommensurable ratio between the units fixed by nature defied even the French Revolution).

The Arabic notation "was brought to perfection in the fifth or sixth century,"* but did not become common in Europe till the sixteenth

*Cajori, *Hist. of Elem. Math.*, p. 154.

century. It is not quite universal yet, the Roman numerals being still used on the dials of time-pieces, in the titles of sovereigns, the numbers of book chapters and subdivisions, and, in general, where an archaic effect is sought. But the Arabic numerals are so much more convenient that they are superseding the Roman in these places. The change has been noticeable even in the last ten or fifteen years.

The extension of the Arabic system to include fractions was made in the latter part of the sixteenth century. But notwithstanding the superior convenience of decimal fractions, they spread but slowly; and it is only in comparatively recent times that they may be said to be more common than "common fractions."

The next step was logarithms—a step taken in 1614. Within the next ten years they were accommodated to what *we* should call "the base" 10.

The dawn of the nineteenth century found decimal coinage well started in the United States, and a general movement toward decimalization under way in France contemporaneous with the political revolution. The subsequent spread of the metric system over most of the continent of Europe and over many other parts of the world has been the means of teaching decimal fractions.

The movement is still on. The value and importance of decimals are now recognized more every year. And much remains to be decimalized. In stock quotations, fractions are not yet expressed decimally. Three great nations have still to adopt decimal weights and measures in popular use, and England has still to adopt decimal coinage. The history of arithmetic has been, in large part, a slow but well-marked growth of the decimal idea.

Those who are working for world-wide decimal coinage, weights and measures—as a time-saver in school-room, counting house and workshop—as a boon that we owe to posterity as well as to ourselves—may learn from such a historical survey both caution and courage. Caution not to expect a sudden change. Multitudes move slowly in matters requiring a mental readjustment. The present reform movements—

for decimal weights and measures in the United States, and decimal weights, measures and coins in Great Britain—are making more rapid progress than the Arabic numerals or decimal fractions made: and the opponents of the present reform are not so numerous or so prejudiced as were their prototypes who opposed the Arabic notation in the Middle Ages and later. Caution also against impatience with a conservatism whose arguments are drawn from the temporary inconvenience of making any change. Courage to work and wait—in line with progress.

In using fractions, the Egyptians and Greeks kept the numerators constant and operated with the denominators. The Romans and Babylonians preferred a constant denominator, and performed operations on the numerator. The Romans reduced their fractions to the common denominator 12, the Babylonians to 60ths. We also reduce our fractions to a common denominator; but we choose 100. One of the most characteristic trends of modern arithmetic is the rapid growth in the use of percentage—another development of the decimal idea. The broker and the biologist, the statistician and the salesman, the manufacturer and the mathematician alike express results in per cents.

These and other changes in the methods of computers have brought about, though tardily, corresponding changes in the subject matter of arithmetic as taught in the schools. Scholastic puzzles are giving place to problems drawn from the life of to-day.

Perhaps one may venture the opinion that, in order to merit a place in the arithmetic curriculum, a topic must be useful either (1) in commerce or (2) in industry or (3) in science. Under (3) may be included, conceivably, a topic whose sole, or chief, use is in later mathematical work. At least two other reasons have been given for retaining a subject: (4) It is required for examination. But it will be found that subjects not clearly justified on one of the grounds above mentioned are rarely required by examining bodies of this generation; and such subjects, if pointed out, would doubtless be withdrawn from any syllabus. (5) It gives superior mental training. But on closer scrutiny this argument becomes somewhat evanescent. A survey of results in that branch of educational psychology which treats of the coefficient of correlation between a pupil's attainments in various activities, weakens one's faith in our ability to give a certain amount of general discipline by a certain amount of special training. Moreover, that discipline can be as well acquired by the study of subjects that serve a direct, useful purpose. We may, then, limit our criteria to these: utility for business or industrial pursuits, and utility for work in science.

Applying these tests to the topics contained in the schoolbooks of a generation ago, we see that many of them are not worthy of a place in the crowded curriculum of our generation. Turning to the schools, we find that many of these topics have, in fact, been dropped. Others are receiving less attention each year. Among such may be mentioned: "true" discount, partnership involving time, and equation of payments (all three giving, besides, a false idea of business), and Troy and apothecaries weight, cube root (except for certain purposes with advanced pupils) and compound proportion.

At the same time, other topics in the arithmetic course are of increasing importance; notably those involving percentage and other decimal operations, and those relating to stock companies and other developments of modern economic activity.

School life is adjusting itself to present social conditions, not only in the topics taught, but in the problems used and the way in which the topics are treated. Good books no longer set problems in stocks involving the purchase of a fractional number of shares!

As Agesilaus, king of Sparta, said, "Let boys study what will be useful to men."

The Greeks studied ἀριθμητική, or theory of numbers, and λογιστική, or practical calculation. Hence the modern definition of arithmetic, "the science of numbers and the art of computation." As Prof. David Eugene Smith points out (in his *Teaching of Elementary Mathematics*) "the modern arithmetic of the schools includes much besides this." It includes the introduction of the pupil to the commercial, industrial and scientific life of to-day on the quantitive side.

Characteristic of our time is the extensive use of arithmetical machines (such as adding machines and instruments from which per cents may be read) and of tables (of square roots for certain scientific work, interest tables for banks, etc.). The initial invention of such appliances is not recent; it is their variety, adaptability and rapidly extending usefulness that may be classed as a present phenomenon.

They have not, however, eliminated the necessity for training good reckoners. They may have narrowed the field somewhat, but in that remaining part which is both practical and necessary they have set the standard of attainment higher. Indeed, an important feature of the present situation is the insistent demand of business men that the schools turn out better computers. There must soon come, in school, a stronger emphasis on accuracy and rapidity in the four fundamental operations.

Emphasis on accuracy and rapidity in calculation leads to the use of "examples" involving abstract numbers. Emphasis on the business applications alone leads to the almost exclusive use of "problems" in which the computative is but an incidental feature. Both are necessary. It has been well said that examples are to the arithmetic pupil what exercises are to the learner on the piano, while problems are to the former what tunes are to the latter. Without exercises, no skill; with exercises alone, no accomplishment. The exercises are for the technique of the art. The teacher can not afford to neglect either.

The last century or more has been the age of special methods in teaching. One has succeeded another in popular favor. Each has taught us an important lesson—something that will be a permanent acquisition to the pedagogy of the science. Few things are more interesting to the student of the history of arithmetic methods than to trace each schoolroom practice of to-day to its origin in some worthy contributor to the science (e. g., in the primary grades, the use of blocks to Trapp, 1780; the "number pictures" to Von Busse; counting by 2's, 3's, ... as preparation for the multiplication tables to Knilling and Tanck; etc.). More recently several famous methods have appeared which are still advocated. But the present trend is toward a choosing of the best from each—an eclectic method.

Most questions of method have never been adequately tested. It is, for instance, asserted by some and denied by others that pupils would know as much arithmetic at the end of the 8th school year if they were to begin arithmetic with the 5th or even later. History may well lead us to doubt the proposition; but who can tell? The greatest desideratum in all arithmetic teaching to-day is the thorough study of the subject by the scientific methods employed in educational psychology. Some one with facilities for doing this service for arithmetic could be a benefactor indeed. Questions that are matters for accurate test and measurement should not always remain questions. Meantime, empiricism is unavoidable.

To summarize the tendencies noted: the decimalization of arithmetic, growth of percentage, elimination of many topics from the school curriculum in arithmetic with increased emphasis on others, modernizing the treatment of remaining topics, demand for more accuracy and rapidity in computation, inclination toward an eclectic method in teaching arithmetic, present empiricism pending scientific investigation. This list is, of course, far from exhaustive, but it is believed to be true and significant.

Lacking such exact information as that just asked for as the desideratum of to-day, we may make the best of mere observation of the trends of our time. And as to the great movements in the history of the art of arithmetic itself, the conclusions are definite and decisive. By orienting ourselves, by studying the past and noting the currents, we may acquaint ourselves with the direction of present forces and may take part in shaping our course. Our to-days are conditioned by our yesterdays, to-morrow by to-day.

For the multiplication of whole numbers the Italians invented many methods.* Pacioli (1494) gives eight. Of these, only one was in common use, and it alone has survived in commerce and the schools. Shown in I on p. 45. It was called *bericuocolo* (honey cake or ginger bread) by the Florentines, and *scacchiera* (chess or checker board) by the Venetians. The little squares in the partial products fell into disuse (and with them the names which they made appropriate) leaving the familiar form II on p. 45. The Treviso arithmetic (1478), the first arithmetic printed, contains a long example in multiplication, which appears about as it would appear on the blackboard of an American school to-day.

In 1585 appeared Simon Stevin's immortal *La Disme*, only seven pages, but the first publication to expound decimal fractions, though the same author had used them in an interest table published the year before. III on p. 45 is from *La Disme*, and shows Stevin's notation (the numbers in circles, or parentheses, indicating the order of decimals, tenths the first order etc.) IV is the same example with the decimals expressed by the notation now prevalent in America. Let us call this arrangement of work Stevin's method.

An arrangement in which all decimal points are in a vertical column (see V below) is said to have been used by Adrian Romain a quarter of a century later. He may not have been the inventor of this arrangement; but, for the sake of a name, call it Romain's method.

Romain's method is advocated in a few of the best recent advanced arithmetics, but Stevin's is still vastly the more common; and these two are the only methods in use. Romain's has four slight advantages: (1) A person setting down an example from dictation can begin to write the multiplier as soon as the place of its decimal point is seen, while in Stevin's method he waits to hear the entire multiplier before he writes any of it, in order to have its last (right-hand) figure stand beneath the

*For the historical facts in this section the author is indebted mainly to Professor Cajori and Prof. David Eugene Smith, the two leading American authorities on the history of mathematics.

last figure of the multiplicand (though this position may be regarded as a non-essential feature in Stevin's arrangement). (2) Romain's method fits more naturally with the "Austrian" method of division (decimal point of quotient over that of dividend). (3) After the partial products are added, it is not necessary to count and point off in the product as many decimal places as there are in the multiplicand and multiplier together, since the decimal point in the product (as well as in the partial products) is directly beneath that in the multiplicand. (4) Romain's method is more readily adapted to abridged multiplication where only approximate results are required. On the other hand, Stevin's method has one very decided advantage: the first figure written in each partial product is directly beneath its figure in the multiplier, so that it is not necessary (as it is in Romain's) to determine the place of the decimal point in a partial product. So important is this, that Stevin's alone has been generally taught to children, notwithstanding the numerous points in favor of Romain's.

It occurred to the writer recently to try to combine in one method the advantages of both of the Flemish methods, and he hit upon the following simple rule:* Write the units figure of the multiplier under the last (right-hand) figure of the multiplicand, begin each partial product (as in the familiar method of Stevin) under the figure by which you are multiplying, and all decimal points in products will then be directly beneath that in the multiplicand. Decimal points in partial products may be written or not, as desired. The reason underlying the rule is apparent. VI shows the arrangement of work.

In this arrangement the placing of the partial products is automatic, as in Stevin's method, and the pointing off in the product is automatic, as in Romain's. It is available for use by the child in his first multiplication of decimals and by the skilled computer in his abridged work.

*Since writing this the author has come upon the same method of multiplication in Lagrange's *Lectures*, delivered in 1795 (p. 29–30 of the Open Court Publishing Co.'s edition). One who invents anything in elementary mathematics is likely to find that "the ancients have stolen his ideas."

To assist in keeping like decimal orders in the same column it is recommended that the vertical line shown in VII and VIII be drawn before the partial products are written. One of the earliest uses of the line as decimal separatrix is in an example in Napier's *Rabdologia* (1617). He draws it through the partial and complete products. It is said to be the first example of abridged multiplication. A circumference is computed whose diameter is 635.

VII illustrates the application of the method here advocated to multiplication in which only an approximation is sought. The diameter of a circle is found by measurement to be 74.28 cm. This is correct to 0.01 cm. No computation can give the circumference to any higher degree of accuracy. Partial products are kept to three places in order to determine the correct figure for the second place in the complete product. The arrangement of work shows what figures to omit.

It should be remarked that all three methods of multiplication of decimals are alike—and like the multiplication of whole numbers—in that one may multiply first by the digit of lowest order in the multiplier or by the digit of highest order first. The method of multiplying by the highest order first was described by the Italian arithmeticians as *a dietro*. Though it may seem to be working *backwards*, it is not so in fact; for it puts the more important before the less, and has practical advantage in abridged multiplication, like that shown in VII. But that question is distinct from the one under consideration.

Stevin writes the last figure of the multiplier under the last figure of the multiplicand; Romain writes units under units; the method here proposed writes units under last. In *whole* numbers, units figure *is* the last.

Applied to the ordinary multiplication of decimals, as in VI or VIII, the method here proposed seems to be well adapted to schoolroom use, possessing all the simplicity of Stevin's. Methods classes in this normal school to whom the method was presented, immediately preferred it, and a grade in the training school used it readily. Of course this proves nothing, for every method is a success in the hands of its advocates.

The changes here set forth are, however, not advocated; they are merely proposed as a possibility.

The analogous method for division of decimals possesses analogous advantages. It avoids the necessity of multiplying the divisor and dividend by such a power of 10 as will make the divisor integral (as in the method now perhaps most in favor) and the necessity of counting to point off in the quotient a number of decimal places equal to the number in the dividend minus that in the divisor (as in the older method still common). Like the latter, it begins the division at once; and like the former, its pointing off is automatic. IX shows the arrangement. The figure under the last figure of the divisor is *units* figure of the quotient. This determines the place of the decimal point. That part of the quotient which projects beyond the divisor, is fractional.

If the order of multiplication used has been *a dietro*, as in VIII, the division in IX is readily seen to be the inverse operation. The partial products appear in the same order as partial dividends.

Like each of the methods in use, it may be abbreviated by writing only the remainders below the dividend. Shown in X.

If the "little castle" method of multiplication of whole numbers, with multiplier above multiplicand, had prevailed, instead of the "chess board," in the fifteenth century, the arrangement now proposed for the multiplication and division of decimals would have afforded slightly greater advantage.

I

				3	2	5	7
				8	9	4	6
			1	9	5	4	2
		1	3	0	2	8	
	2	9	3	1	3		
2	6	0	5	6			
2	9	1	3	7	1	2	2

II

				3	2	5	7
				8	9	4	6
			1	9	5	4	2
		1	3	0	2	8	
	2	9	3	1	3		
2	6	0	5	6			
2	9	1	3	7	1	2	2

III

$$\begin{array}{cccccccc}
 & & & & & (0) & (1) & (2) \\
 & & & & 3 & 2 & 5 & 7 \\
 & & & & 8 & 9 & 4 & 6 \\
\cline{4-8}
 & & & 1 & 9 & 5 & 4 & 2 \\
 & & 1 & 3 & 0 & 2 & 8 & \\
 & 2 & 9 & 3 & 1 & 3 & & \\
2 & 6 & 0 & 5 & 6 & & & \\
\hline
2 & 9 & 1 & 3 & 7 & 1 & 2 & 2 \\
 & & & (0) & (1) & (2) & (3) & (4)
\end{array}$$

IV

$$\begin{array}{cccccccc}
 & & & & 3 & 2. & 5 & 7 \\
 & & & & 8 & 9. & 4 & 6 \\
\cline{4-8}
 & & & 1 & 9 & 5 & 4 & 2 \\
 & & 1 & 3 & 0 & 2 & 8 & \\
 & 2 & 9 & 3 & 1 & 3 & & \\
2 & 6 & 0 & 5 & 6 & & & \\
\hline
2 & 9 & 1 & 3. & 7 & 1 & 2 & 2
\end{array}$$

V

$$\begin{array}{cccccccc}
 & & 3 & 2. & 5 & 7 & & \\
 & & 8 & 9. & 4 & 6 & & \\
\cline{3-8}
 & & & 1. & 9 & 5 & 4 & 2 \\
 & & 1 & 3. & 0 & 2 & 8 & \\
 & 2 & 9 & 3. & 1 & 3 & & \\
2 & 6 & 0 & 5. & 6 & & & \\
\hline
2 & 9 & 1 & 3. & 7 & 1 & 2 & 2
\end{array}$$

VI

$$\begin{array}{cccccccc}
 & & 3 & 2. & 5 & 7 & & \\
 & & & & 8 & 9. & 4 & 6 \\
\cline{4-8}
 & & & 1. & 9 & 5 & 4 & 2 \\
 & & 1 & 3. & 0 & 2 & 8 & \\
 & 2 & 9 & 3. & 1 & 3 & & \\
2 & 6 & 0 & 5. & 6 & & & \\
\hline
2 & 9 & 1 & 3. & 7 & 1 & 2 & 2
\end{array}$$

VII

$$\begin{array}{ccc|cccccc}
 & 7 & 4. & 2 & 8 & & & & \\
 & & & & 3. & 1 & 4 & 1 & 6 \\
\hline
2 & 2 & 2 & 8 & 4 & & & & \\
 & & 7 & 4 & 2 & 8 & & & \\
 & & 2 & 9 & 7 & 1 & & & \\
 & & & & 7 & 4 & & & \\
 & & & & 4 & 5 & & & \\
\cline{1-5}
2 & 3 & 3 & 3 & 6 & & & &
\end{array}$$

VIII

$$\begin{array}{cccc|cccc}
 & & 3 & 2. & 5 & 7 & & \\
 & & & & 8 & 9. & 4 & 6 \\
\hline
2 & 6 & 0 & 5 & 6 & & & \\
 & 2 & 9 & 3 & 1 & 3 & & \\
 & & 1 & 3 & 0 & 2 & 8 & \\
 & & & 1 & 9 & 5 & 4 & 2 \\
\hline
2 & 9 & 1 & 3 & 7 & 1 & 2 & 2
\end{array}$$

IX

		3	2.	5	7		
				8	9.	4	6
2	9	1	3.	7	1	2	2
2	6	0	5	6			
	3	0	8	1	1		
	2	9	3	1	3		
		1	4	9	8	2	
		1	3	0	2	8	
			1	9	5	4	2
			1	9	5	4	2

X

		3	2.	5	7		
				8	9.	4	6
2	9	1	3.	7	1	2	2
	3	0	8	1	1		
		1	4	9	8	2	
			1	9	5	4	2

The invention of printing was important for arithmetic, not only because it made books more accessible, but also because it spread the use of the Hindu ("Arabic") numerals with their decimal notation.

The oldest text-book on arithmetic to use these numerals is said to be that of Avicenna, an Arabian physician of Bokhara, about 1000 A. D. (Brooks). According to Cardan (sixteenth century) it was Leonardo of Pisa who introduced the numerals into Europe (by his *Liber Abaci*, 1202). In England, though there is one instance of their use in a manuscript of 1282, and another in 1325, their use is somewhat exceptional even in the fifteenth century. Then came printed books and a more general acceptance of the decimal notation.

The importance of this step can hardly be over-estimated. Even the Greeks, with all their mathematical acumen, had contented themselves with mystic and philosophic properties of numbers and had made comparatively little progress in the art of computation. They lacked a suitable notation. When such a notation was adopted at the close of the Middle Ages, the art advanced rapidly. That advance was one feature of the Renaissance, a detail in the great intellectual awakening of that marvelous half century from 1450 to 1500, "the age of progress".

The choice between the old and the new in arithmetical notation is well pictured by the illustration* of arithmetic in the first printed cyclopedia, the *Margarita Philosophica* (1503). Two accountants are at their tables. The old man is using the abacus; the young man, the Hindu numerals so familiar to us. The aged reckoner looks askance at his youthful rival, in whose face is hope and confidence; while on a dais behind both stands the goddess to decide which shall have the ascendency. Her eyes are fixed on the younger candidate, at her right, and there can be no doubt that to the new numerals is to be the victory. The background of the picture is characteristically medieval. It is an apt illustration of the passing of the old arithmetic. To us of four

*See frontispiece.

centuries after, it whispers (as one has said of the towers of old Oxford) "the last enchantment of the Middle Age."

The anonymous book known as the Treviso arithmetic, from its place of publication, is the first arithmetic ever printed. It appeared in 1478. In this Italian work of long ago the multiplication looks modern. But long division was by the galley (or "scratch") method then prevalent.

Pacioli's *Summa di Arithmetica* was published in 1494 (some say ten years earlier). It also uses the Hindu numerals.*

Tonstall's arithmetic (1522) was "the first important arithmetical work of English authorship."† De Morgan calls the book "decidedly the most classical which ever was written on the subject in Latin, both in purity of style and goodness of matter."

Recorde's celebrated *Grounde of Artes* (1540) was written in English. It uses the Hindu numerals, but teaches reckoning by counters. The exposition is in dialogue form.

The first English work on double entry book-keeping, by John Mellis (London, 1588), has an appendix on arithmetic.

The Pathway to Knowledge, anonymous, translated from Dutch into English by W. P., was published in London in 1596. It contains two lines which are immortal. The translator has been said to be the author of the lines. In modernized form they are known to every schoolboy. Of all the arithmetical doggerel of that age, this is pre-eminently the classic:

> "Thirtie daies hath September, Aprill, June, and November,
> Februarie, eight and twentie alone; all the rest thirtie and one."

On the subject of early arithmetics De Morgan's *Arithmetical Books* is the standard work. An interesting contribution to the subject is Prof.

*In Pacioli's work, the words "zero" (*cero*) and "million" (*millione*) are found for the first time in print. Cantor, II, 284.

†Cajori, *Hist. of Elem. Math.*, p. 180.

David Eugene Smith's illustrated article, "The Old and the New Arithmetic," published by Ginn & Co. in their Textbook Bulletin, February, 1905.

NAPIER'S RODS AND OTHER MECHANICAL AIDS TO CALCULATION.

No mathematical invention to facilitate computation has been made for three centuries that is comparable to logarithms. Napier's rods, or "Napier's bones," once famous, owe their interest now largely to the fact that they are the invention of the man who gave logarithms to the world, John Napier, baron of Merchiston. The inventor's description of the rods is contained in his *Rabdologia*, published in 1617, the year of his death.

The rods consist of 10 strips of wood or other material, with square ends. A rod has on each of its four lateral faces the multiples of one of the digits. One of the rods has, on the four faces respectively, the multiples of 0, 1, 9, 8; another, of 0, 2, 9, 7; etc. Each square gives the product of two digits, the two figures of the product being separated by the diagonal of the square. E. g., in Fig. 2 the lowest right hand square contains the digits 7 and 2, 72 being the product of 9 (at the left of the same row) and 8 (at the top of the rod).

Fig. 2 represents the faces of the rods giving the multiples of 4, 3, and 8, placed together and against a rod containing the nine digits to be used as multiplier, all in position to multiply 438 by any number—say 26. The products are written off, from the rods. But the tens digit in each case is to be added to the next units digit; that is, the two figures in a rhomboid are to be added. The operation of multiplying 438 by 26, after arranging the rods as in Figure 2, would be somewhat as follows: beginning at the right hand and multiplying first by 6, we have 8, $4 + 8$, (carrying the 1) $1 + 1 + 4$, 2, giving the number (from left to right) 2628, the first partial product. Similarly 876 is read from the row of squares at the right of the multiplier 2. It is shifted one place to the left in writing it under the former partial product. Then these two numbers are added.

Somewhat analogous is the use of the rods for division.

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†From Lucas, III, 76.

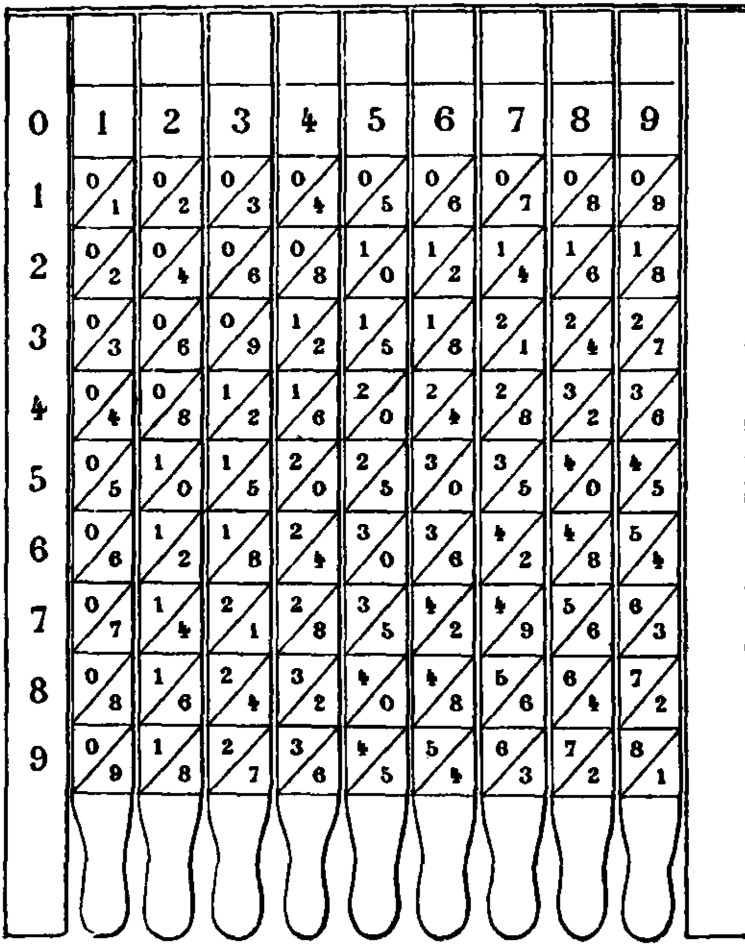

Fig. 1.†

"It is evident that they would be of little use to any one who knew the multiplication table as far as 9×9."‡ Though published (and invented) later than logarithms, which we so much admire, the rods were welcomed more cordially by contemporaries. Several editions of the

‡Dr. Glaisher in his article "Napier" in the *Encyclopædia Britannica.*

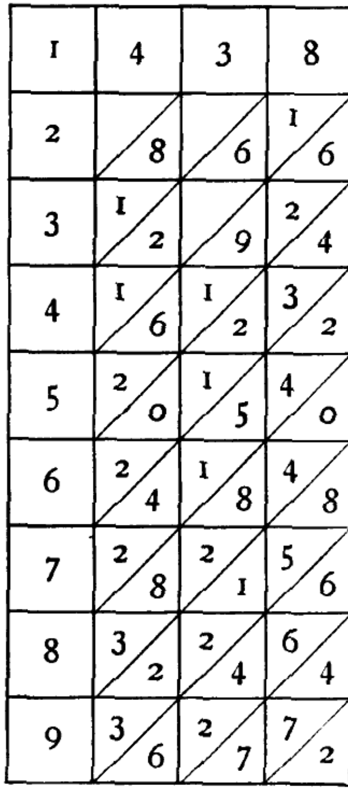

Fig. 2.

Rabdologia were brought out on the Continent within a decade. "Nothing shows more clearly the rude state of arithmetical knowledge at the beginning of the seventeenth century than the universal satisfaction with which Napier's invention was welcomed by all classes and regarded as a real aid to calculation."‡ It is from this point of view that the study of the rods is interesting and instructive to us.

The *Rabdologia* contains other matter besides the description of rods for multiplication and division. But such mechanical aids to calculation are soon superseded.

It is worthy of note in this connection, however, that in the absence of so facile an instrument for calculation as our Arabic notation, simple mechanical devices might be found so serviceable as to persist for centuries. The abacus, which is familiar to almost every one, but only as a historical relic, a piece of illustrative apparatus, or a toy, was a highly important aid to computation among the Greeks and Romans. Similar to the abacus is the Chinese *swan pan.* It is said that Oriental accountants are able, by its use, to make computations rivaling in accuracy and speed those performed by Occidentals with numerals on paper.

Modern adding machines, per cent devices, and the more complicated and costly calculating instruments have led up to such mechanical marvels as "electrical calculating machines" and the machines of Babbage and Scheutz, which latter prepare tables of logarithms and of logarithmic functions without error arithmetical or typographical, computing, stereotyping and delivering them ready for the press.

If Napier's rods be regarded as exemplars of such products of the nineteenth century, they are primitive members of a long line of honorable succession.

AXIOMS IN ELEMENTARY ALGEBRA.

Many text-books on the subject introduce equations with a list of axioms such as the following:

1. Things equal to the same thing or equal things are equal to each other.
2. If equals be added to equals, the sums are equal.
3. If equals be subtracted from equals, the remainders are equal.
4. If equals be multiplied by equals, the products are equal.
5. If equals be divided by equals, the quotients are equal.
6. The whole is greater than any of its parts.
7. Like powers, or like roots, of equals are equal.

These time-honored "common notions" are the foundation of logical arithmetic. On them is based also the reasoning of algebra. But it is most desirable that, when we extend their meaning to the comparison of algebraic numbers, we should notice the limitations of the axioms. Generalization is a characteristic of mathematics. When we generalize, we remove limitations that have been stated or implied. A proposition true with those limitations may or may not be true without them. For illustration: When we proceed from geometry of two dimensions to geometry of three dimensions, the limitation, always understood in plane geometry, that all figures considered are (except while employing the motion postulate for superposition) in the plane of the paper or blackboard, is removed. Some of the propositions true in plane geometry hold also in solid, and some do not. Compare in this respect the two theorems, "Through a given *external* point only one perpendicular can be drawn to a given line," and, "Through a given *internal* point only one perpendicular can be drawn to a given line."* For another illustration see the paragraph (p. 24), "Are there more than one set of prime factors of a number?" *No* when factor means arithmetic number; *yes* when the meaning of the word is extended to include complex numbers. See also instances of the "fallacy of accident," p. 67 f.

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*Using the term *perpendicular* in the sense customary in elementary geometry.

We might expect that some of the axioms of arithmetic would need qualification when we attempt to extend them so as to apply to algebraic numbers. And that is what we find. But we do not find that all authors have notified their readers of the limitations or have observed them in their own use of the axioms. Surely it is not too much to expect that the axioms of a science shall be true and applicable *in the sense in which the terms are used in that science.*

The fifth, or "division axiom," should receive the important qualification given it by the best of the books, "divided by equals, *except zero.*" Without such limitation the statement is far from axiomatic.

A writer of the sixth "axiom" may also have on another page something like this: "+3 is the whole, or *sum.*" Seeing that one of its parts is +7, one wonders how the author, in a text-book on algebra, could ever have written the "axiom," "The whole is greater than any of its parts."

$$\begin{array}{r} +7 \\ -5 \\ +2 \\ -1 \\ \hline +3 \end{array}$$

In the seventh axiom, like roots of equals are equal *arithmetically.* Otherwise worded: Like real roots of equals are equal, like signs being taken.*

*The defense often heard for the unqualified axiom, Like roots of equals are equal, in algebra—that *like* here means *equal*—would reduce the axiom to a platitude, Roots are equal if they are equal. Besides being insipid, this is insufficient. To be of any use, the axiom must mean, that if C and D are known to represent each a square root, or each a cube root, of A and B respectively, and if A and B are known to be equal, then C and D are as certainly known to be two expressions for the same number. Now in the case of square roots this inference is justified only when like signs are taken. For cube roots, if A = B = 1, then $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ is a cube root of A, and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ is a cube root of B; but $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ are not expressions for the same number. If their modulus (page 76) be taken as

When we use the word "equal" in the axioms, do we mean anything else than "same"—If two numbers are the same as a third number, they are the same as each other, etc.?

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their absolute value, they are equal to each other and to the real cube root 1 in absolute value. If our axiom be made to read, Like odd *real* roots are equal, it is applicable to such roots without trouble. A has but one cube root that is real, and B has but one, and they are equal.

It is interesting to notice in passing that the two numbers just used, $-\frac{1}{2}+\frac{1}{2}\sqrt{-3}$ and $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$, are a pair of unequal numbers each of which is the square of the other.

DO THE AXIOMS APPLY TO EQUATIONS?

Most text-books in elementary algebra use them as if they applied. Most of the algebras have, somewhere in the first fifty or sixty pages, something like this:

$$3x + 4 = 19.$$

Subtracting 4 from each member,

$$3x = 15. \qquad \text{Ax. 3}$$

Dividing by 3,

$$x = 5. \qquad \text{Ax. 5}$$

This shows how common some very loose thinking on this subject is. For although no mistake has been made in the algebraic operation, the citation of axioms as authority for these steps opens the way for a pupil to divide both members of an equation by an unknown, in which case he drops a solution,* or to apply one of the other axioms and introduce a solution.

As a matter of fact, the axioms do not apply directly to equations: for (A) one can follow the axioms, make no mistake, and arrive at a result which is incorrect: (B) he can violate the axioms and come out right: (C) the axioms, from their very nature, can not apply directly to equations.

(A) *To follow axioms and come out wrong:*

$$x - 1 = 2. \qquad (1)$$

Multiplying each member by $x - 5$,

$$x^2 - 6x + 5 = 2x - 10. \qquad \text{Ax. 4}$$

*Every teacher of elementary algebra is aware of the tendency of pupils (unless carefully guided) to "divide through by x" when possible, and to fail to note that they have lost out the solution $x = 0$.

Subtracting $x - 7$ from each member,

$$x^2 - 7x + 12 = x - 3. \qquad \text{Ax. 3}$$

Dividing each member by $x - 3$,

$$x - 4 = 1. \qquad \text{Ax. 5}$$

Adding 4 to each member,

$$x = 5. \qquad \text{Ax. 2}$$

But $x = 5$ does not satisfy (1). The only value of x that satisfies (1) is 3.

Misunderstanding at this point is so common that it is deemed best to be explicit at the risk of being tedious. The multiplication by $x - 5$ introduces the solution $x = 5$, and the division by $x-3$ loses the solution $x = 3$. Now it may be argued, that the axioms of the preceding section when properly qualified exclude division by zero, and that $x - 3$ is here a form of zero since 3 is the value of x for which equation (1) is true. Exactly; but this only shows that in operating with equations the question for what value of x they are true is bound to be raised. The attempt to qualify the axioms and adjust them to this necessity will, if thoroughgoing, lead to principles of equivalency of equations.* Any objector is requested to study carefully the principles of equivalency as set forth in one of the best algebras and notice their relation to the axioms on the one hand and to operations with equations on the other,

*Such, for example, as the following:

To add or subtract the same expression (known or unknown) to both members of an equation, does not affect the value of x (the resulting equation is equivalent to the original).

To multiply or divide both members by a known number not zero, does not affect the value of x.

To multiply or divide both members by an integral function of x, introduces or loses, respectively, solutions (namely, the solution of the equation formed by putting the multiplier equal to zero) it being understood that the equations are in the standard form.

and see whether he is not then prepared to say that the axioms do not apply *directly* to equations.

It should be noted that the foregoing is not an attack on the integrity of the axioms, but only on the application of them where they are not applicable.

If it be objected that in (A) the axioms are not really followed, the reply is, that they are here followed as they are naturally followed by pupils taught to apply them directly to equations, and as they are occasionally followed by the authors of some elementary algebras, only the errors are here made more glaring and the process reduced *ad absurdum.*

(B) *To violate the axioms and come out right:*

In order to avoid the objection that the errors made by violating two axioms may just balance each other, only *one* axiom will be violated.

$$x - 1 = 2. \qquad (1)$$

Add 10 to one member *and not to the other.* This will doubtless be deemed a sufficiently flagrant transgression of the "addition axiom":

$$x + 9 = 2. \qquad (2)$$

Multiplying each member by $x - 3$,

$$x^2 + 6x - 27 = 2x - 6. \qquad (3) \quad \text{Ax. 4}$$

Subtracting $2x - 6$ from each member,

$$x^2 + 4x - 21 = 0. \qquad (4) \quad \text{Ax. 3}$$

Dividing each member by $x + 7$,

$$x - 3 = 0. \qquad (5) \quad \text{Ax. 5}$$

Adding 3 to each member,

$$x = 3. \qquad \text{Ax. 2}$$

Inasmuch as 3 is *the correct root* of equation (1), the error in the first step must have been balanced by another, or by several. It was done in obtaining (3) and (5), though at both steps the axioms were applied.

(C) *The axioms, from their very nature, can not have any direct application to equations.*

The axioms say that—if equals be added to equals etc.—the results are equal. But the question in solving equations is, For what value of x are they equal? Of course they are equal for *some* value of x. So when something was added to one member and not to the other, the results were equal *for some value of* x. Arithmetic, dealing with numbers, needs to know that certain resulting numbers are equal to certain others; but algebra, dealing with the equation, the conditional equality of expressions, needs to know on *what condition* the expressions represent the same number—in other words, for what values of the unknown the equation is true. In (B) above, the objection to equation (2) is not that its two members are not equal (they are "equal" as much as are the two members of the first equation) but that they are not equal *for the same value of* x as in the first equation; that is (2) is not *equivalent* to (1).

The principles of equivalency of equations as given in a few of the best of the texts are not too difficult for the beginner. The *proof* of them may well be deferred till later. Even if never proved, they would be, for the present purpose, vastly superior to axioms that do not apply. To give *no* reasons would be preferable to the practice of quoting axioms that do not apply.

The axioms have their place in connection with equations; namely, in the proof of the principles of equivalency. To apply the axioms directly in the solution of equations is an error.

Pupils can hardly be expected to think clearly about the nature of the equation when they are so misled. How the authors of the great majority of the elementary texts can have made so palpable a mistake in so elementary a matter, is one of the seven wonders of algebra.

CHECKING THE SOLUTION OF AN EQUATION.

The habit which many high-school pupils have of checking their solution of an equation by first substituting for x in both members of the given equation, performing like operations upon both members until a numerical identity is obtained, and then declaring their work "proved," may be illustrated by the following "proof," in which the absurdity is apparent:

$$1 + \sqrt{x+2} = 1 - \sqrt{12 - x}. \qquad (1)$$

Solution:

$$\sqrt{x+2} = -\sqrt{12-x} \qquad (2)$$
$$x + 2 = 12 - x \qquad (3)$$
$$2x = 10$$
$$x = 5.$$

"Proof":

$$1 + \sqrt{5+2} = 1 - \sqrt{12-5}$$
$$\sqrt{5+2} = -\sqrt{12-5}$$
$$5 + 2 = 12 - 5$$
$$7 = 7.$$

Checking in the legitimate manner—by substituting in one member of the given equation and reducing the resulting number to its simplest form, then substituting in the other member and reducing to simplest form—we have $1+\sqrt{7}$ for the first member, and $1-\sqrt{7}$ for the second. As these are not equal numbers, 5 is not a root of the equation. There is no root.

The 5 was introduced in squaring. That is, $x = 5$ satisfies equation (3) but not (2) or (1). By the change of a sign in either (1) or (2) we obtain an equation that is true for $x = 5$:

$$1 + \sqrt{x+2} = 1 - \sqrt{12 - x}.$$

When rational equations are derived from irrational by involution, there are always other irrational equations, differing from these in the sign of a term, from which the same rational equations would be derived.

In a popular algebra may be found the equation

$$x + 5 - \sqrt{x+5} = 6$$

and in the answer list printed in the book, "4, or -1" is given for this equation. 4 is a solution, but -1 is not. Unfortunately this instance is not unique.

As the fallacy in the erroneous method shown above is in assuming that all operations are reversible, that method may be caricatured by the old absurdity,

To prove that

$$5 = 1.$$

Subtracting 3 from each,

$$2 = -2.$$

Squaring

$$4 = 4.$$

$$\therefore\ 5 = 1!$$

ALGEBRAIC FALLACIES.

A humorist maintained that in all literature there are really only a few jokes with many variations, and proceeded to give a classification into which all jests could be placed—a limited list of type jokes. A fellow humorist proceeded to reduce this number (to three, if the writer's memory is correct). Whereupon a third representative of the profession took the remaining step and declared that there are none. Whether these gentlemen succeeded in eliminating jokes altogether or in adding another to an already enormous number, depends perhaps on the point of view.

The writer purposes to classify and illustrate some of the commoner algebraic fallacies, in the hope, not of adding a striking original specimen, but rather of standardizing certain types, at the risk of blighting them. Fallacies, like ghosts, are not fond of light. Analysis is perilous to all species of the genus.

Of the classes, or subclasses, into which Aristotle divided the fallacies of logic, only a few merit special notice here. Prominent among these is that variety of paralogism known as undistributed middle. In mathematics it masks as the fallacy of converse, or employing a process that is not uniquely reversible as if it were. For example, the following:*

Let c be the arithmetic mean between two *unequal* numbers a and b; that is, let

$$a + b = 2c. \qquad (1)$$

Then

$$(a+b)(a-b) = 2c(a-b)$$
$$a^2 - b^2 = 2ac - 2bc.$$

*Taken, with several of the other illustrations, from the fallacies compiled by W. W. R. Ball. See his *Mathematical Recreations and Essays.*

Transposing,

$$a^2 - 2ac = b^2 - 2bc. \tag{2}$$

Adding c^2 to each,

$$a^2 - 2ac + c^2 = b^2 - 2bc + c^2 \tag{3}$$
$$\therefore\ a - c = b - c$$

and

$$a = b$$

But a and b were taken unequal.

Of course the two members of (3) are arithmetically equal but of opposite quality; their squares, the two members of (2), are equal. The fallacy here is so apparent that it would seem superfluous to expose it, were it not so common in one form or another.

For another example take the absurdity used in the preceding section to caricature an erroneous method of checking a solution of an equation. Let us resort to a parallel column arrangement:

A bird is an animal;	Two equal numbers have equal squares;
A horse is an animal;	These two numbers have equal squares;
∴ A horse is a bird.	∴ These two numbers are equal.
The untutored man pooh-poohs at this, because the *conclusion* is absurd, but fails to notice a like fallacy on the lips of the political speaker of his own party.	The first-year high-school pupil derides this whenever the *conclusion* is absurd, but would allow to pass unchallenged the fallacious method of checking shown in the preceding section.

In case of indicated square roots the fallacy may be much less apparent. By the common convention as to sign, $+$ is understood before $\sqrt{\ }$. Considering, then, only the positive even root or the real odd root, it is true that "like roots of equals are equal," and

$$\sqrt[n]{ab} = \sqrt[n]{a} \cdot \sqrt[n]{b}.$$

But if a and b are negative, and n even, the identity no longer holds, and by assuming it we have the absurdity

$$\begin{aligned} \sqrt{(-1)(-1)} &= \sqrt{-1} \cdot \sqrt{-1} \\ \sqrt{1} &= (\sqrt{-1})^2 \\ 1 &= -1. \end{aligned}$$

Or take for granted that $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$ for all values of the letters. The following is an identity, since each member $= \sqrt{-1}$:

$$\sqrt{\frac{1}{-1}} = \sqrt{\frac{-1}{1}}.$$

Hence!

$$\frac{\sqrt{1}}{\sqrt{-1}} = \frac{\sqrt{-1}}{\sqrt{1}}.$$

Clearing of fractions,

$$(\sqrt{1})^2 = (\sqrt{-1})^2.$$

Or

$$1 = -1.$$

The "fallacy of accident," by which one argues from a general rule to a special case where some circumstance renders the rule inapplicable, and its converse fallacy, and De Morgan's suggested third variety of the fallacy, from one special case to another, all find exemplification in pseudo-algebra. As a general rule, if equals be divided by equals, the quotients are equal; but not if the equal divisors are any form of zero. The application of the general rule to this special case is the method underlying the largest number of the common algebraic fallacies.

$$x^2 - x^2 = x^2 - x^2.$$

Factoring the first member as the difference of squares, and the second by taking out a common factor,

$$(x + x)(x - x) = x(x - x). \qquad (1)$$

Canceling $x - x$,

$$x + x = x$$

$$(2)$$

$$2x = x$$

$$2 = 1. \qquad (3)$$

Dividing by 0 changes identity (1) into equation (2), which is true for only one value of x, namely 0. Dividing (2) by x leaves the absurdity (3).

Take another old illustration:*

Let

$$x = 1.$$

Then

$$x^2 = x.$$

And

$$x^2 - 1 = x - 1.$$

---

*Referred to by De Morgan as "old" in a number of the *Athenæum* of forty years ago.

Dividing both by $x - 1$,

$$x + 1 = 1.$$

But

$$x = 1.$$

Whence, by substituting,

$$2 = 1.$$

The use of a divergent series furnishes another type of fallacy, in which one assumes something to be true of all series which in fact is true only of the convergent. For this purpose the harmonic series is perhaps oftenest employed.

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots.$$

Group the terms thus:

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots \text{ to 8 terms}\right) + \left(\frac{1}{17} + \dots \text{ to 16 terms}\right) + \dots.$$

Every term (after the second) in the series as now written $> \frac{1}{2}$. Therefore the sum of the first n terms increases without limit as n increases indefinitely.* The series has no finite sum; it is divergent. But if the signs in this series are alternately $+$ and $-$, the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

*The sum of the first 2^n terms $> 1 + \frac{1}{2}\,n$.

is convergent. With this in mind, the following fallacy is transparent enough:

$$\begin{aligned}\log 2 &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots \\ &= \left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) - \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right) \\ &= \left[\left(1 + \frac{1}{3} + \frac{1}{5} + \dots\right) + \left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right)\right] \\ &\qquad - 2\left(\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \dots\right) \\ &= \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots\right) \\ &= 0.\end{aligned}$$

But

$$\log 1 = 0.$$

Suppose ∞ written in place of each parenthesis.

∞ and 0 are both convenient "quantities" for the fallacy maker.

By tacitly assuming that all real numbers have logarithms and that they are amenable to the same laws as the logarithms of arithmetic numbers, another type of fallacy emerges:

$$(-1)^2 = 1.$$

Since the logarithms of equals are equal,

$$\begin{gathered}2\log(-1) = \log 1 = 0 \\ \therefore \log(-1) = 0 \\ \therefore \log(-1) = \log 1 \\ \text{and } -1 = 1.\end{gathered}$$

The idea of this type is credited to John Bernoulli. Some great minds have turned out conceits like these as by-products, and many amateurs have found delight in the same occupation. To those who enjoy weaving a mathematical tangle for their friends to unravel, the diversion may be recommended as harmless. And the following may be suggested as promising points around which to weave a snarl: the tangent of an angle becoming a discontinuous function for those particular values of the angle which are represented by $(n + \frac{1}{2})\pi$; discontinuous algebraic functions; the fact that when h, j, and k are rectangular unit vectors the commutative law does not hold, but $hjk = -kjh$; the well-known theorems of plane geometry that are not true in solid geometry without qualification; etc.

Let us use one of these to make a fallacy to order. In the fraction $1/x$, if the denominator be diminished, the fraction is increased.

When $x = 5, 3, 1, -1, -3, -5$, a decreasing series, then $1/x = 1/5$, $1/3$, 1, -1, $-1/3$, $-1/5$, an increasing series, as, by rule, each term of the second series is greater than the term before it: $1/3 > 1/5$, $1 > 1/3$, $-1/5 > -1/3$. Then the fourth term is greater than the third; that is $-1 > +1$.

Neither the fallacies of formal logic nor those of algebra invalidate sound reasoning. From the counterfeit coin one does not infer that the genuine is valueless. Scrutiny of the counterfeit may enable us to avoid being deceived later by some particularly clever specimen. Counterfeit coins also, if so stamped, make good playthings.

TWO HIGHEST COMMON FACTORS.

If asked for the H. C. F. of $a^2 - x^2$ and $x^3 - a^3$, one pupil will give $a - x$, and another $x - a$. Which is right? Both. It is only in such a case that pupils raise the question; but the example is not peculiar in having two H.C.F. If the given expressions had been $x^2 - a^2$ and $x^3 - a^3$, $x - a$ would naturally be obtained, and would probably be the only H.C.F. offered; but $a - x$ is as much a common factor and is of as high a degree. Perhaps it is taken as a matter of course—certainly it is but rarely stated—that every set of algebraic expressions has two highest common factors, arithmetically equal but of opposite quality.

As the term "highest" is used in a technical way, the purist will perhaps pardon the solecism "two highest."

Similarly, of course, there are two L.C.M. of every set of algebraic expressions. By going through the answer list for exercises in L.C.M. in an algebra and changing the signs, one obtains another list of answers.

POSITIVE AND NEGATIVE NUMBERS.

To speak of arithmetical numbers as positive, is still so common an error as to need correction at every opportunity. The numbers of arithmetic are not positive. They are numbers *without quality.* Negatives are not later than positives, either in the individual's conception or in that of the race. How can the idea of one of two *opposites* be earlier than the other, or clearer? The terms "positive" and "negative" being correlative, neither can have meaning without the other.*

An "algebraic balance" has been patented and put on the market,† designed to illustrate positive and negative numbers, also transposition and the other operations on an equation. It is composed of a system of levers and scale pans with weights. The value of this excellent apparatus in illustrating positive and negative numbers is in showing them to be opposites of each other. E. g., a weight in the positive scale pan neutralizes the pull on the beam exerted by a weight of equal mass in the negative scale pan. The two weights are of equal mass, as the two numbers are of equal arithmetical value. When the weight is put into *either* scale pan, it represents a "real," or quality, number; it becomes either $+$ or $-$.

The unfortunate expression "less than nothing"(due to Stifel), the attempt to consider negative numbers apart from positive and to teach negative after positive, and the name "fictitious" for negative numbers, all seem absurd enough now; but they became so only when the real significance of positive and negative as opposites was clearly seen. The value of the illustration from debts and credits (due to the Hindus)

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*A good exercise to develop clear thinking as to the relation between positive, negative and arithmetic numbers is, to consider the correspondence of the positive and negative solutions of an equation to the arithmetic solutions of the problem that gave rise to the equation, and the question to what primary assumptions this correspondence is due.

†By F. C. Donecker, Chicago. Described in *School Science and Mathematics.* See also "Another Algebraic Balance," by N. J. Lennes, *id.*, Nov. 1905; and "Content-Problems for High School Algebra," by G. W. Meyers, *id.*, Jan. 1907, reprinted from *School Review.*

and from the thermometer, lies in the aptness for bringing out the oppositeness of positive and negative.

For the illustration from directed lines, see Fig. 3 on the following page.

It is appropriate that the advertisements of the algebraic balance use the quotation from Cajori's *History of Elementary Mathematics*: "Negative numbers appeared 'absurd' or 'fictitious' so long as mathematicians had not hit upon a *visual or graphical representation of them. . . .* Omit all illustrations by lines, or by the thermometer, and negative numbers will be as absurd to modern students as they were to the early algebraists."

If the sect OR, one unit long and extending to the right of O, be taken to represent $+1$, then -1 will be represented by OL, extending one to the left of O. $+a$ would be pictured by a line a units long and to the right; $-a$, a units long and to the left. This simplest and best-known use of directed lines gives us a geometric representation of real numbers. The Hindus early gave this interpretation to numbers of opposite quality; but it does not appear to have been given by a European until 1629, by Girard.*

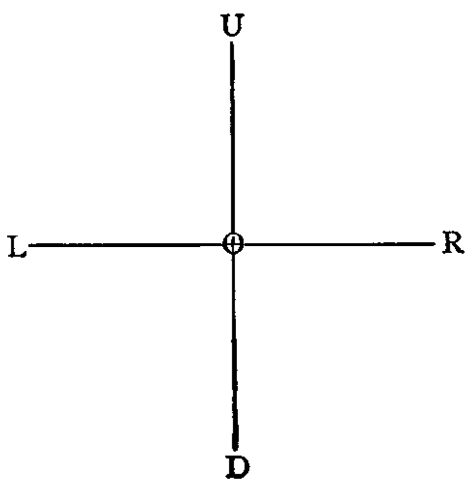

Fig. 3.

Conceiving the line of unit length to be revolved in what is assumed as the positive direction (counter-clockwise) -1 may be called the factor that revolves from OR ($+1$) to OL (-1). Then $\sqrt{-1}$ is the factor which, being used *twice*, produces that result; using it *once* as a factor revolving the line through one of the two right angles. Then OU pictures the number $+\sqrt{-1}$. Similarly, since multiplication of -1

*Albert Girard, *Invention Nouvelle en l'Algèbre*, Amsterdam. Perhaps also the first to distinctly recognize imaginary roots of an equation.

by $-\sqrt{-1}$ twice produces $+1$, $-\sqrt{-1}$ may be considered as the factor which revolves from OL through one right angle to OD. If distances to the right are called $+$, then distances to the left are $-$, and $+\sqrt{-1} \cdot b$ denotes a line b units long and extending up, and $-\sqrt{-1} \cdot b$ a line b units

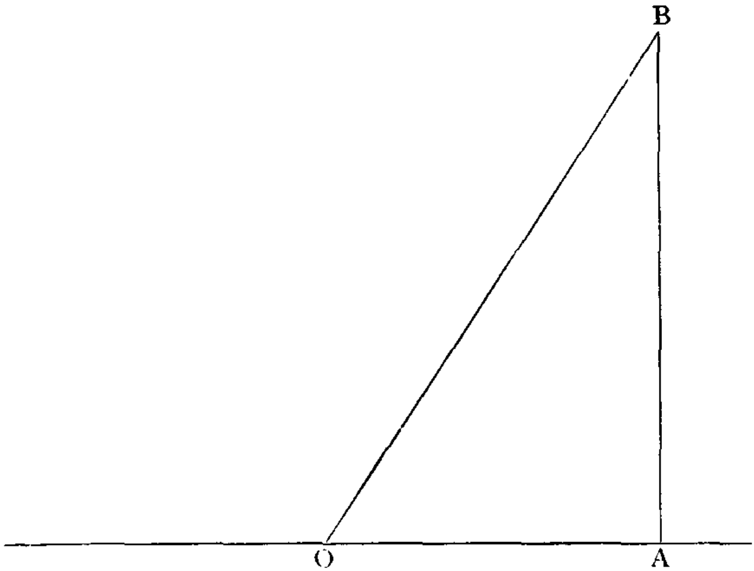

Fig. 4.

long extending down. The geometric interpretation of the imaginary was made by H. Kühn in 1750, in the *Transactions of the St. Petersburg Academy.*

To represent graphically the number $a + b\sqrt{-1}$ (see Fig. 4), we lay off OA in the $+$ direction and a units long; AB, b units long and in the direction indicated by $\sqrt{-1}$; and draw OB. The directed line OB represents the complex number $a + b\sqrt{-1}$. And the length of OB, $\sqrt{a^2 + b^2}$, is the *modulus* of $a + b\sqrt{-1}$. The geometric interpretation of such a number was made by Jean Robert Argand, of Geneva, in his *Essai*, 1806. The term "modulus" in this connection was first used by him, in 1814.

These geometric interpretations by Kühn and Argand, and especially one made by Wessel,* who extended the method to a representation in space of three dimensions, may be regarded as precursors of the beautiful methods of vector analysis given to the world by Sir William Rowan Hamilton in 1852 and 1866 under the name "quaternions."

The letter i as symbol for the unit of imaginary numbers, $\sqrt{-1}$, was suggested by Euler. It remained for Gauss to popularize the sign i and the geometric interpretations made by Kühn and Argand.

The contrasting terms "real" and "imaginary" as applied to the roots of an equation were first used by Descartes. The name "imaginary" was so well started that it still persists, and seems likely to do so, although it has long been seen to be a misnomer.† A few writers use the terms *scalar* and *orthotomic* in place of *real* and *imaginary.*

The historical development of this subject furnishes an illustration of the general rule, that, as we advance, each new generalization includes as special cases what we have previously known on the subject. The general form of the complex number, $a + bi$, includes as special cases the real number and the imaginary. If $b = 0$, $a + bi$ is real. If $a = 0$, $a + bi$ is imaginary. The common form of a complex number is the sum of a real number and an imaginary.‡

In 1799 Gauss published the first of his three proofs that every algebraic equation has a root of the form $a + bi$.

The linear equation forces us to the consideration of numbers of opposite quality: $x - a = 0$ and $x + a = 0$, satisfied by the values $+a$ and $-a$ respectively. The pure quadratic gives imaginary in contrast with real roots: $x^2 - a^2 = 0$ and $x^2 + a^2 = 0$ satisfied by $\pm a$ and $\pm ai$.

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*To the Copenhagen Academy of Sciences, 1797.

†It is interesting to notice the prestige of Descartes's usage in fixing the language of algebra: the first letters of the alphabet for knowns, the last letters for unknowns, the present form of exponents, the dot between factors for multiplication.

‡Professor Schubert (p. 14) adds that "we have found the most general numerical form to which the laws of arithmetic can lead."

The complete quadratic

$$ax^2 + bx + c = 0$$

has for its roots a pair of conjugate complex numbers when the discriminant, $b^2 - 4ac$, is negative and b is not $= 0$.

But though the recognition of imaginary and complex numbers is a necessary consequence of simple algebraic analysis, no complete understanding or appreciation of them is possible until there is some tangible or visible representation of them. History's lesson to us in this respect is plain: positive and negative, imaginary, and complex numbers must be graphically represented in teaching algebra.

The algebraic balance mentioned on page 72 might be further developed by the addition of an appliance whereby imaginary numbers should be illustrated, a weight put into a certain pan having the effect of pulling the main beam to one side, and arrangements for pulling the beam in several other directions to illustrate complex numbers.

If in a football game we denote the forces exerted in the direction OR (in Fig. 3) by positive real numbers, then the opponents' energy exerted in exactly the opposite direction, OL, will be denoted by negative numbers. Forces in the line of OU or OD will be denoted by imaginary numbers; and all other forces in the game, acting in any other direction on the field, will be denoted by complex numbers of the general type.*

Each force represented by a general complex number is resolvable into two forces, one represented by a real number and the other by an imaginary, as OB (in Fig. 4) is the resultant of OA and AB.

A trigonometric representation of an imaginary number as exponent is furnished by the formula

$$e^i = \cos 1 + i \sin 1.$$

*Illustration from Taylor's *Elements of Algebra*, where the visual representation of imaginary and complex numbers is made in full.

ILLUSTRATIONS OF THE LAW OF SIGNS IN ALGEBRAIC MULTIPLICATION.

A Geometric Illustration.

If distances to the right of O be called +, then distances to the left will be −. Call distances up from O +, and those down −. Rectangle OR has ab units of area. *Assume* that the product ab is +.

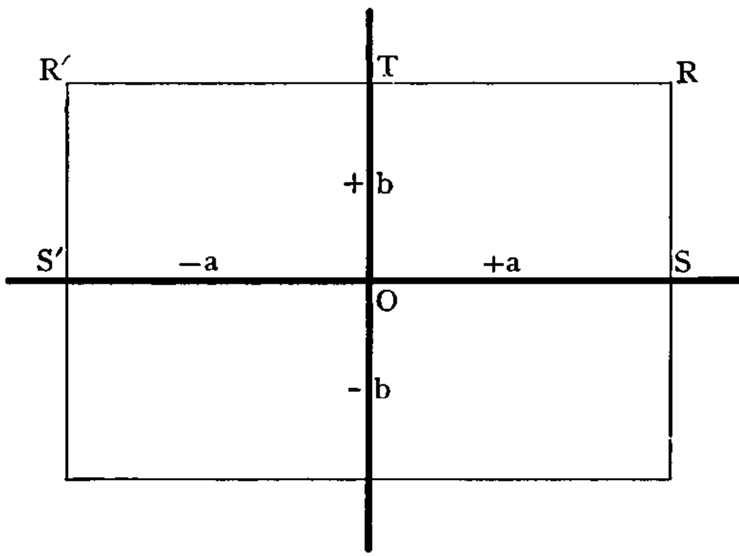

Fig. 5.

Suppose SR to move to the left until it is a units to the left of O, in the position S′R′. The base diminished, became zero, and passed through that value, and therefore is now negative; so also the rectangle. The product of $-a$ and $+b$ is $-ab$.

Suppose TR′ to move downward until it is b units below O. The rectangle, previously −, has passed through zero, and must now be +. The product of $-a$ and $-b$ is $+ab$.

Similarly $(+a)(-b) = -ab$.

From a Definition of Multiplication.

Multiplication is the process of performing upon one of two given numbers (the multiplicand) the same operation which is performed upon the primary unit to obtain the other number (the multiplier.)*

When the multiplier is an arithmetical integer, the primary unit is that of arithmetic, 1, and we have the special case that is correctly defined in the primary school as, "taking one number as many times as there are units in another."

Suppose we are to multiply +4 by +3. *Assuming* +1 as the primary unit, the multiplier is produced by taking that unit additively "three times," $(+1)+(+1)+(+1)$. That is what the number +3 means; and to multiply +4 by it, means to do that to +4. $(+4)+(+4)+(+4) = +12$. Similarly, the product of -4 by $+3 = (-4)+(-4)+(-4) = -12$.

To multiply +4 by -3: The multiplier is the result obtained by taking three times additively the primary unit *with its quality changed.* The product of +4 by -3 is therefore the result obtained by taking three

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*In this definition, "the same operation which is performed upon the primary unit to obtain the multiplier" is to be understood to mean the most fundamental operation by which the multiplier may be obtained from unity, or that operation which is primarily signified by the multiplier. E. g., If the multiplier is 2, this number primarily means unity taken twice, or the unit added to itself; multiplying 4 by 2 therefore means adding 4 to itself, giving the result 8. Dr. Young, in his new book, *The Teaching of Mathematics*, p. 227, says that as 2 is $1+1^2$, therefore 2×4 would by this definition be $4+4^2$, or 20; or, as 2 is $1+1/1$, therefore 2×4 would be $4+4/4$, or 5; etc. But while it is true that $1+1^2$ and $1+1/1$ are each equal to 2, neither of them is the primary signification of 2, or represents 2 in the sense of the definition. Neither of them is a proper statement of the multiplier "within the meaning of the law."

It is not maintained that this definition has no difficulties, or that it directly helps a learner in comprehending the meaning of such a multiplication as $\sqrt{2}\times\sqrt{3}$, but only that it is a generalization that is helpful for the purpose for which it is used, and that it is in line with the fundamental idea of multiplication so far as that idea is understood.

The definition is only tentative, and this treatment does not pretend to be a proof.

times additively $+4$ *with its quality changed.* $(-4)+(-4)+(-4) = -12$. Similarly, to multiply -4 by -3 is to take three times additively -4 with its quality changed: $(+4) + (+4) + (+4) = +12$.

Summarizing the four cases, we have "the law of signs": the product is $+$ when the factors are of like quality, $-$ when they are of unlike quality.

A more General Form of the Law of Signs.

In deriving the law from the definition of multiplication, the primary unit was assumed as $+1$. Assume -1 as the primary unit, and multiply $+4$ by $+3$. The multiplier, $+3$, is obtained from the primary unit, -1, by taking three times additively the unit with its sign changed. Performing the same operation on the multiplicand, $+4$, we have $(-4)+(-4)+(-4) = -12$. Similarly, the product of -4 by $+3 = (+4) + (+4) + (+4) = +12$. To multiply $+4$ by -3: The multiplier is the result obtained by taking three times additively the unit, -1, without change of sign; therefore the product of $+4$ by $-3 = (+4) + (+4) + (+4) = +12$. So also -4 multiplied by -3 gives -12. Summarizing *these* four cases, we have the law of signs when -1 is taken as the primary unit: the product is $-$ when the factors are of like quality, $+$ when they are of unlike quality.

In the geometric illustration above, we first assumed the rectangle $+a$ by $+b$ to be $+$. Assuming the contrary, the sign of each subsequent product is reversed, and we have a geometric illustration of the law of signs when -1 is taken as the primary unit.

The law of signs taking $+1$ as the primary unit, and that taking -1 as the primary unit, may be combined into one law thus: If the two factors are alike in quality, the product is like the primary unit in quality; if the two factors are opposite in quality, the quality of the product is opposite to that of the primary unit. Or: *Like* signs give *like* (like the primary unit); *unlike* signs give *unlike* (the unit).

The assumption of still other numbers as primary unit leads to other laws—other "algebras."

Multiplication as a Proportion.

Since by definition a product bears the same relation to the multiplicand that the multiplier bears to the primary unit, this equality of relation may be stated in the form of a proportion:

product : multiplicand :: multiplier : primary unit

or,

primary unit : multiplier :: multiplicand : product.

Gradual Generalization of Multiplication.

From the time when Pacioli found it necessary (and difficult) to explain how, in the case of proper fractions in arithmetic, the product is less than the multiplicand, to the present with its use of the term *multiplication* in higher mathematics, is a long evolution. It is one of the best illustrations of the generalization of a term that was etymologically restricted at the beginning.

EXPONENTS.

The definition of *exponent* found in the elementary algebras is sufficient for the case to which it is applied—the case in which the exponents are arithmetic integers. Our assumption of a primary unit for algebra being what it is, the distinction between arithmetic numbers as exponents and positive numbers as exponents is usually neglected. Or we may simply define positive exponent. The meaning of negative and fractional exponents is easily deduced. In fact those who first used exponents and invented an exponential notation (Oresme in the fourteenth century and Stevin independently in the sixteenth) had fractions as well as whole numbers as exponents. And negative exponents had been invented before Wallis studied them in the seventeenth century. Each of these can be defined separately. And modern mathematics has used other forms of exponents. They have been made to follow the laws of exponents first proved for ordinary integral exponents, and their significance has been assigned in conformity thereto. Each separate species of exponent is defined. A unifying conception of them all might express itself in a definition covering all known forms as special cases. The general treatment of exponents is yet to come.

WANTED: A DEFINITION OF EXPONENT that shall be general for elementary mathematics.

AN EXPONENTIAL EQUATION.

The chain-letters, once so numerous, are now—it is to be hoped—obsolete. In the form that was probably most common, the first writer sends three letters, each numbered 1. Each recipient is to copy and send three, numbered 2, and so on until number 50 is reached.

Query: If every one were to do as requested, and it were possible to avoid sending to any person twice, what number of letter would be reached when every man, woman and child in the world should have received a letter?

Let n represent the number. Take the population of the earth to be fifteen hundred million. Then this large number is the sum of the series

$$3, \quad 3^2, \quad 3^3, \; \ldots \quad 3^n$$

$$\text{S} = \frac{a(r^n - 1)}{r - 1} = \frac{3(3^n - 1)}{2}$$

$$\frac{3}{2}(3^n - 1) = 1{,}500{,}000{,}000$$

$$3^n - 1 = 1{,}000{,}000{,}000$$

$$n \log 3 = \log(10^9)$$

$$n = \frac{9}{\log 3} = 18.86.$$

There are not enough people in the world for the letters numbered 19 to be all sent.

TWO NEGATIVE CONCLUSIONS REACHED IN THE NINETEENTH CENTURY.

1. That general equations above the fourth degree are insoluble by pure algebra.

The solution of equations of the third and fourth degree had been known since 1545. Two centuries and a half later, young Gauss, in his thesis for the doctorate, proved that every algebraic equation has a root, real or imaginary.* He made the conjecture in 1801 that it might be impossible to solve by radicals any general equation of higher degree than the fourth. This was proved by Abel, a Norwegian, whose proof was printed in 1824, when he was about twenty-two years old. Two years later the proof was published in an expanded form, with more detail.

Thus inventive effort was turned in other directions.

2. That the "parallel postulate" of Euclid can never be proved from the other postulates and axioms.

Ever since Ptolemy, in the second century, the attempt had been made to prove this postulate, or "axiom," and thus place it among theorems. In 1826, Lobachevsky, professor and rector at the University of Kasan, Russia, proved the futility of the attempt, and published his proof in 1829. He constructed a self-consistent geometry in which the other postulates and axioms are assumed and the contrary of this, thus showing that this is independent of them and therefore can not be proved from them. No notice of his researches appeared in Germany till 1840. In 1891 Lobachevsky's work was made easily available to English readers through a translation by Prof. George Bruce Halsted.†

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*Of this proof, published when Gauss was twenty-two years old, Professor Maxime Bôcher remarks (*Bulletin of Amer. Mathematical Society*, Dec. 1904, p. 118, noted): "Gauss's first proof (1799) that every algebraic equation has a root gives a striking example of the use of intuition in what was intended as an absolutely rigorous proof by one of the greatest and at the same time most critical mathematical minds the world has ever seen." It should be added that Gauss afterward gave two other proofs of the theorem.

†Austin, Texas, 1892. It contains a most interesting introduction by the trans-

The effort previously expended in attempting the impossible was henceforth to be turned to the development of non-Euclidean geometry, to investigating the consequences of assuming the contrary of certain axioms, to n-dimensional geometry. "As is usual in every marked intellectual advance, every existing difficulty removed has opened up new fields of research, new tendencies of thought and methods of investigation, and consequently new and more difficult problems calling for solution."*

High-school geometry must simply *assume* (choose) Euclid's postulate of parallels, perhaps preferably in Playfair's form of it: Two intersecting lines can not both be parallel to the same line.

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lator. Dr. Halsted translated also Bolyai's work (1891), compiled a *Bibliography of Hyperspace and Non-Euclidean Geometry* (1878) of 174 titles by 81 authors, and has himself written extensively on the subject, being probably the foremost writer in America on non-Euclidean geometry and allied topics.

*Withers, p. 63–4.

In contrast to Euclid's postulate (just quoted) Lobachevsky's is, that through a given point an indefinite number of lines can be drawn in a plane, none of which cut a given line in the plane, while Riemann's postulate is, that through the point no line can be drawn in the plane that will not cut the given line. Thus we have three elementary plane geometries.

An excellent simple illustration of the contrast has been devised: Let AB and PC be two straight lines in the same plane, both unlimited

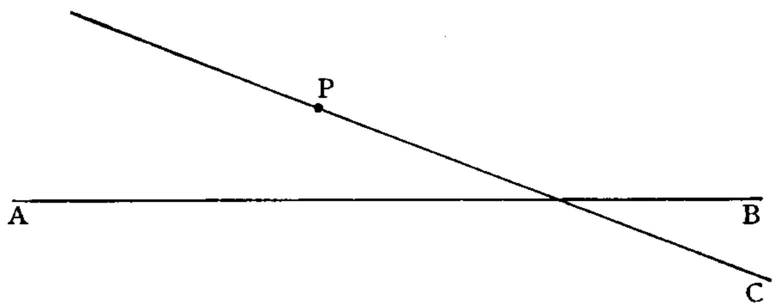

Fig. 6.

in both directions; AB fixed in position; and PC rotating about the point P, say in the positive (counter-clockwise) direction, intersecting first toward the right as shown in Figure 6.

"Three different results are logically possible. When the rotating line ceases to intersect the fixed line in one direction [toward the right] it will immediately intersect in the opposite direction [toward the left], or it will continue to rotate for a time before intersection takes place, or else there will be a period of time during which the two lines intersect in both directions. The first of these possibilities gives Euclid's, the second Lobachevsky's, and the third Riemann's geometry.

"The mind's attitude toward these three possibilities taken successively illustrates in a curious way the essentially empirical nature of the straight line as we conceive it. Logically one of these possibilities is just as acceptable as the other. From this point of view strictly taken there is certainly no reason for preferring one of them to another. Psychologically, however, Riemann's hypothesis seems absolutely contradictory, and even Euclid's is not quite so acceptable as that of Lobachevsky."

As a slight test of the relative acceptability of these hypotheses to the unsophisticated mind, the present writer drew on the blackboard a figure like that above, mentioned in simple language the three possibilities, and asked pupils to express opinion on slips of paper. Forty-six out of 54 voted that the second is the true one. Two said they "guessed" it is, twenty-one "thought" so, thirteen "felt sure," and ten "knew." Six "thought" that the first supposition is correct, and two "felt sure" of it. No one voted for the third, and the writer has never heard but one person express opinion in favor of the third supposition. Some of the pupils had had a few weeks of plane geometry. Of these, most who voted in the majority wanted to change as soon as it was pointed out that the second supposition implies that two intersecting lines can both be parallel to the same line. Undoubtedly some of the more immature were unable to grasp the idea that the lines are of unlimited length, and possibly it may be somewhat general that those who favor the second supposition do not fully grasp that idea. Such a test merely illustrates that Euclid's postulate is not in all its forms apodictic.

The whole question of parallel postulates is admirably treated by Dr. Withers,* to whose book (p. 117) the writer is indebted for the two paragraphs quoted above.

In trigonometry. The familiar figure in trigonometry representing the line values of the tangent of an angle at the center of a unit circle as the angle increases and passes through 90° is another form of this

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*John William Withers, *Euclid's Parallel Postulate: Its Nature, Validity, and Place in Geometrical Systems*, his thesis for the doctorate at Yale, published by The Open Court Publishing Co., 1905. It includes a bibliography of about 140 titles on this and more or less closely related subjects, mentioning Halsted's bibliography of 174 titles and Roberto Bonola's of 353 titles. To these lists might be added Manning's *Non-Euclidean Geometry* (1901) which is brief, elementary and interesting.

figure. And the assumption that intersection of the final (revolving) side with the line of tangents begins at an infinite distance below at the instant it ceases above, places our trigonometry on a Euclidean basis.

Parallels meet at infinity. Kepler's definition would seem paradoxical if offered in elementary geometry, but is valuable in more advanced work, and is intelligible enough when made in the language of limits. Let PP′ be perpendicular to SQ; let Q move farther and farther to the right while P remains fixed; and let P′PR be the limit toward which angle P′PQ approaches as the distance of Q from P′ increases without limit.* Then PR is parallel to SQ. That is, parallelism is attributed to the limiting position of intersecting lines as the point of intersection recedes without limit; which, for the sake of brevity, we may express by the familiar sentence, "Parallels meet at infinity."

The three postulates again. Now suppose PS to move, P remaining fixed and S moving to the left, TPP′ being the limit of angle SPP′ as P′S increases without limit. Then PT is parallel to SQ. According to Euclid's postulate PT and PR are one straight line; according to

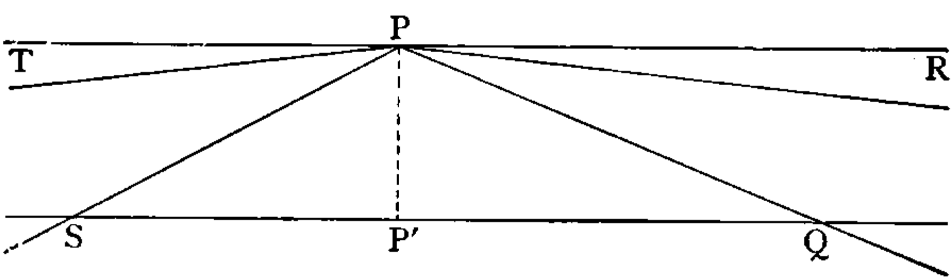

Fig. 7.

Lobachevsky's they are not; while according to Riemann's Q and S can

*In Fig. 6 the moving line rotated until after it ceased to intersect the fixed line toward the right. In the present illustration (Fig. 7) PQ rotates only as Q, the point of intersection, recedes along the line SP′Q.

not recede to an infinite distance (but Q comes around, so to speak, through S, to P′ again) and there is no limiting position (in the terminology of the theory of limits) and no parallel in the Euclidean sense of the term.

"A rectangular hole 13 inches long and 5 inches wide was discovered in the bottom of a ship. The ship's carpenter had only one piece of board with which to make repairs, and that was but 8 inches square (64 square inches) while the hole contained 65 square inches. But he

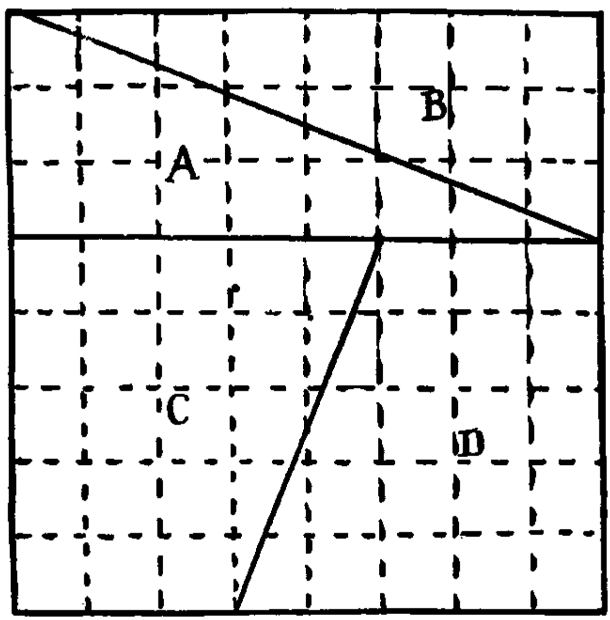

Fig. 8.

knew how to cut the board so as to make it fill the hole"! Or in more prosaic form:

Fig. 8 is a square 8 units on a side, area 64; cut it through the heavy lines and rearrange the pieces as indicated by the letters in Fig. 9, and you have a rectangle 5 by 13, area 65. Explain.

Fig. 10 explains. EH is a straight line, and HG is a straight line, but they are not parts of the same straight line. Proof:

Let X be the point at which EH produced meets GJ; then from the

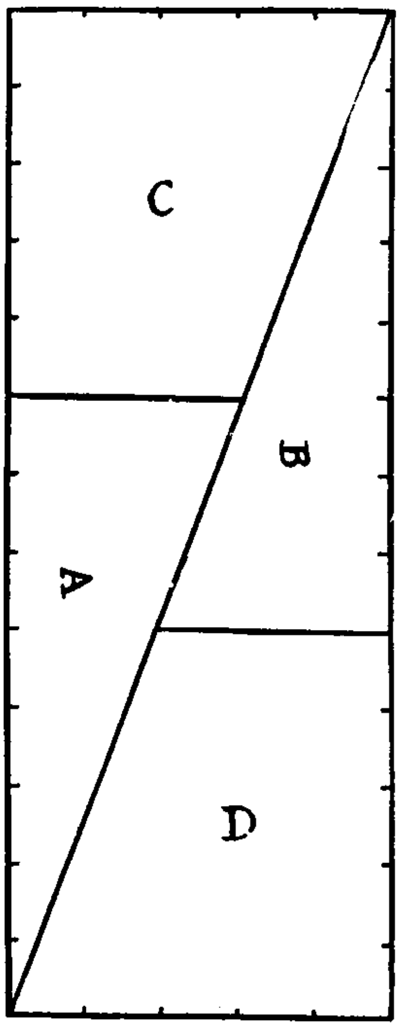

Fig. 9.

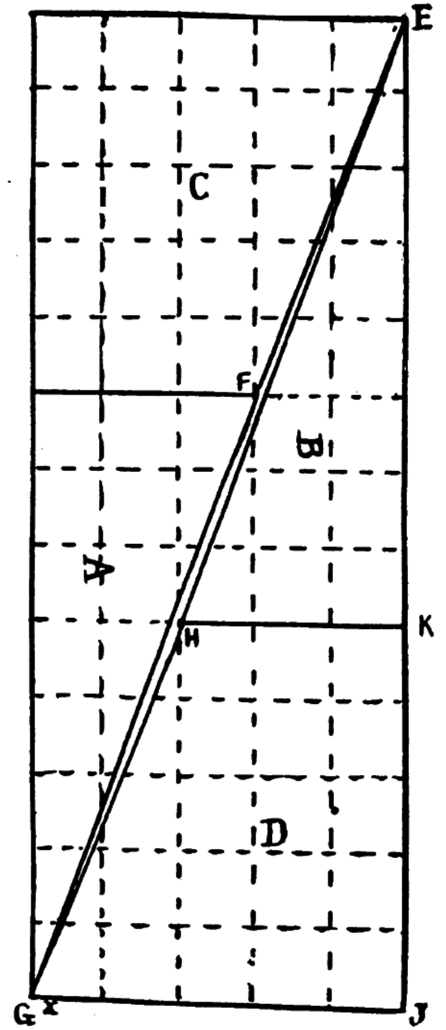

Fig. 10.

similarity of triangles EHK and EXJ

$$XJ : HK = EJ : EK$$

$$XJ : 3 = 13 : 8$$

$$XJ = 4.875.$$

But

$$GJ = 5.$$

Similarly, EFG is a broken line.

The area of the rectangle is, indeed, 65, but the area of the rhomboid EFGH is 1.

Professor Ball* uses this to illustrate that proofs by dissection and superposition are to be regarded with suspicion until supplemented by mathematical reasoning.

"This geometrical paradox ... seems to have been well known in 1868, as it was published that year in Schlömilch's *Zeitschrift für Mathematik und Physik*, Vol. 13, p. 162."

In an article in *The Open Court*, August 1907, (from which the preceding four lines are quoted), Mr. Escott generalizes this puzzle. The puzzle is so famous that his analysis can not but be of interest. With his permission it is here reproduced:

In Fig. 11, it is shown how we can arrange the same pieces so as to form the three figures, A, B, and C. If we take $x = 5$, $y = 3$, we shall have A = 63, B = 64, C = 65.

Let us investigate the three figures by algebra.

$$\begin{aligned}
A &= 2xy + 2xy + y(2y - x) = 3xy + 2y^2 \\
B &= (x + y)^2 = x^2 + 2xy + y^2 \\
C &= x(2x + y) = 2x^2 + xy \\
C - B &= x^2 - xy - y^2 \\
B - A &= x^2 - xy - y^2.
\end{aligned}$$

These three figures would be equal if $x^2 - xy - y^2 = 0$, i. e., if

$$\frac{x}{y} = \frac{1 + \sqrt{5}}{2}$$

* *Recreations*, p. 49.

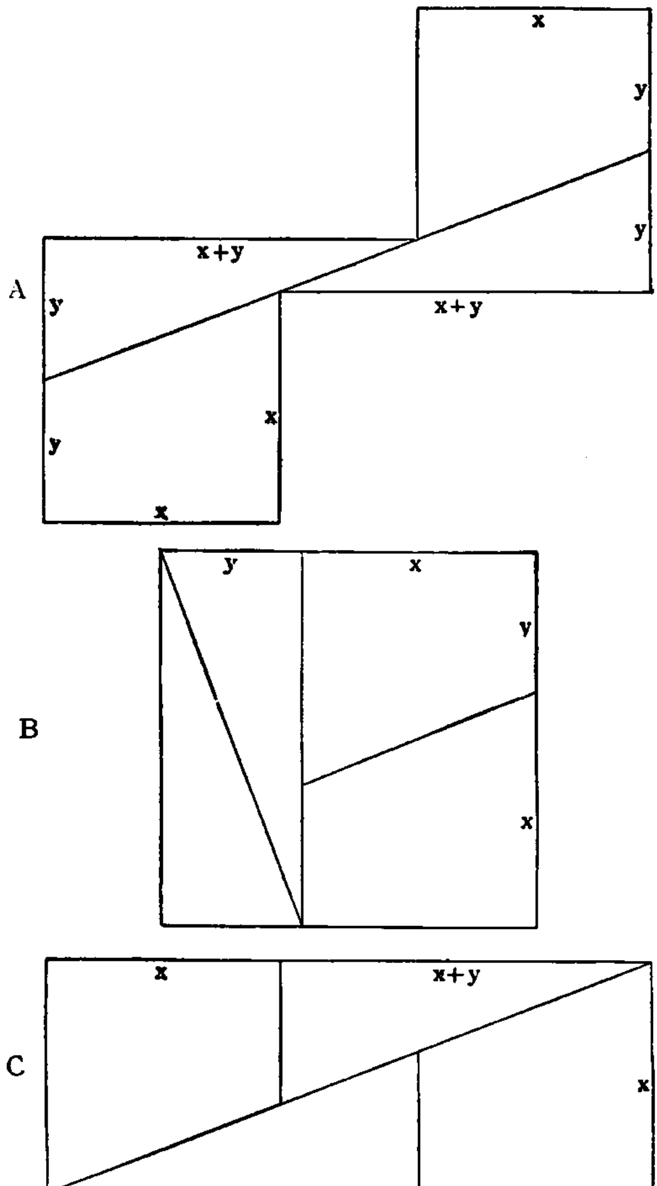

Fig. 11.

so the three figures cannot be made equal if x and y are expressed in rational numbers.

We will try to find rational values of x and y which will make the difference between A and B or between B and C unity.

Solving the equation

$$x^2 - xy - y^2 = \pm 1$$

we find by the Theory of Numbers that the y and x may be taken as any two consecutive numbers in the series

$$1, 2, 3, 5, 8, 13, 21, 34, 55, \dots$$

where each number is the sum of the two preceding numbers.

The values $y = 3$ and $x = 5$ are the ones commonly given. For these we have, as stated above, $\mathrm{A} < \mathrm{B} < \mathrm{C}$.

The next pair, $x = 8$, $y = 5$ give $\mathrm{A} > \mathrm{B} > \mathrm{C}$, i. e., $\mathrm{A} = 170$, $\mathrm{B} = 169$, $\mathrm{C} = 168$.

Fig. 12 shows an interesting modification of the puzzle.

$$\begin{aligned}
\mathrm{A} &= 4xy + (y + x)(2y - x) = 2y^2 + 2yz + 3xy - xz \\
\mathrm{B} &= (x + y + z)^2 = x^2 + y^2 + z^2 + 2yz + 2zx + 2xy \\
\mathrm{C} &= (x + 2z)(2x + y + z) = 2x^2 + 2z^2 + 2yz + 5zx + xy.
\end{aligned}$$

When $x = 6$, $y = 5$, $z = 1$ we have $\mathrm{A} = \mathrm{B} = \mathrm{C} = 144$.

When $x = 10$, $y = 10$, $z = 3$ we have $\mathrm{A} > \mathrm{B} > \mathrm{C}$, viz.,

$$\mathrm{A} = 530,\ \mathrm{B} = 529,\ \mathrm{C} = 528.$$

Another puzzle is made by constructing a cardboard rectangle 13 by 11, cutting it through one of the diagonals (Fig. 13) and sliding one triangle against the other along their common hypotenuse to the position shown in Fig. 14. Query: How can Fig. 14 be made up of square VRXS, 12 units on a side, area 144, + triangle PQR, area 0.5,

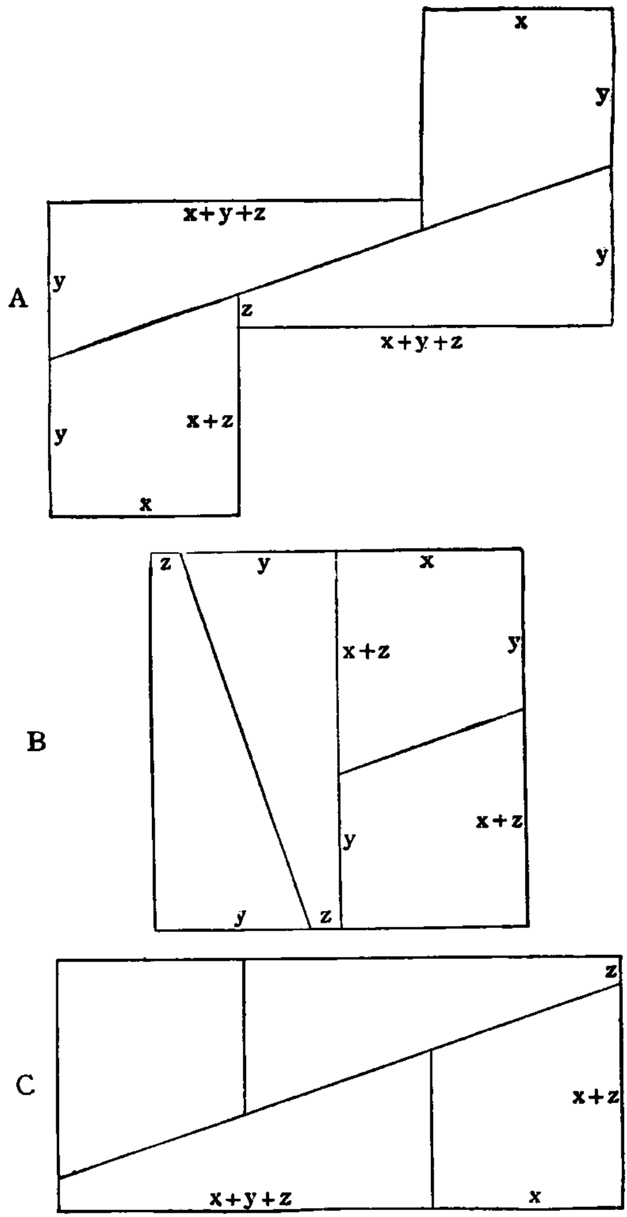

Fig. 12.

+ triangle STU, area 0.5, = total area 145; when the area of Fig. 13 is only 143?

Inspection of the figures, especially if aided by the cross lines, will show that VRXS is not a square. VS is 12 long; but SX < 12. TX $= 11$ (the shorter side in Fig. 13) but ST < 1 (see ST in Fig. 13).

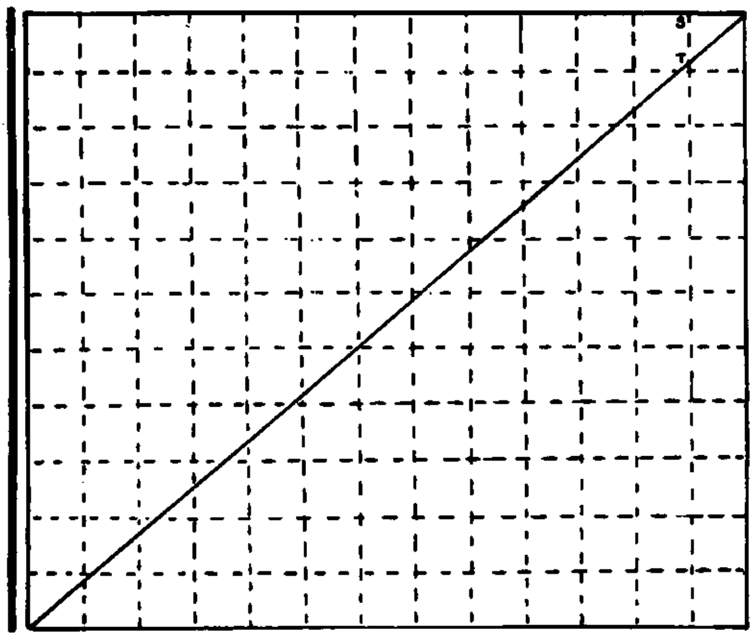

Fig. 13.

$$\begin{aligned} \text{ST} : \text{VP} &= \text{SU} : \text{VU} \\ \text{ST} : 11 &= 1 : 13 \\ \text{ST} &= \tfrac{11}{13}. \\ \text{Rectangle VRXS} &= 12 \times 11\tfrac{11}{13} = 142\tfrac{2}{13}. \\ \text{Triangle PQR} &= \text{triangle STU} = \tfrac{1}{2} \cdot \tfrac{11}{13} \cdot 1 = \tfrac{11}{26}. \\ \text{Fig. 14} &= \text{rectangle} + 2 \text{ triangles} \\ &= 142\tfrac{2}{13} + \tfrac{11}{13} = 143. \end{aligned}$$

By sliding the triangles one place (to the first cross line) in the other direction we appear to have a rectangle 14 by 10 and two small triangles with an area of $\frac{1}{2}$ each, total area 141—as much smaller than Fig. 13 as Fig. 14 is larger. Slide the triangles one more place in the direction last used, and the apparent area is 139. The explanation is of course

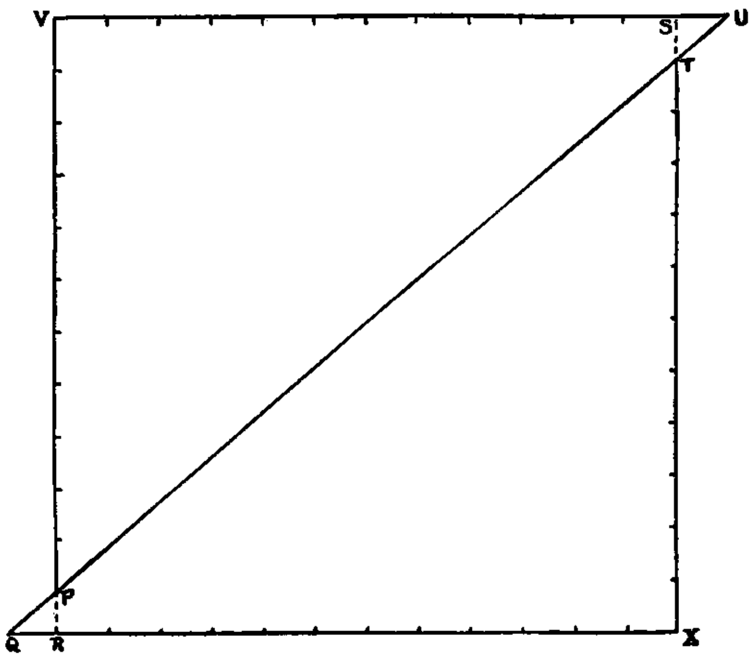

Fig. 14.

similar to that given for Fig. 14.

This paradox also might be treated by an analysis resembling that by which Mr. Escott has treated the preceding.

Very similar is a puzzle due to S. Loyd, "the puzzlist." Fig. A is a square 8×8, area 64. Fig. B shows the pieces rearranged in a rectangle apparently 7×9, area 63.

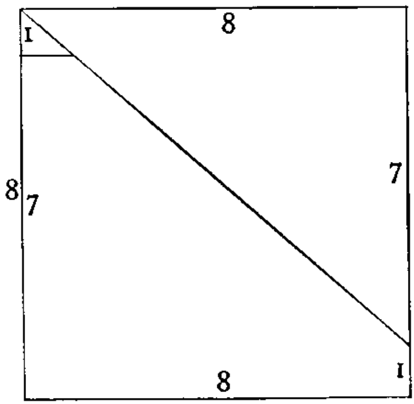

Fig. A.

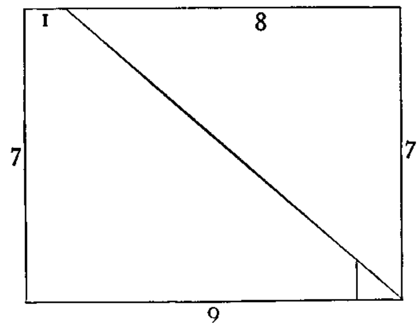

Fig. B.

*Paradromic rings.** A puzzle of a very different sort is made as follows. Take a strip of paper, say half as wide and twice as long as this page; give one end a half turn and paste it to the other end. The ring thus formed is used in theory of functions to illustrate a surface that has only one face: a line can be drawn on the paper from any point of it to any other point of it, whether the two points were on the same side or on opposite sides of the strip from which the ring was made. The ring is to be slit—cut lengthwise all the way around, making the strip of half the present width. State in advance what will result. Try and see. Now predict the effect of a second and a third slitting.

*The theory of these rings is due to Listing, *Topologie*, part 10. See Ball's *Recreations*, p. 75–6.

The theorem seems to have been pleasing to the ancients, as it is to high-school pupils to-day, that a plane surface can be divided into equilateral triangles, squares, or regular hexagons, and that these are the only regular polygons into which the surface is divisible. As a regular hexagon is divided by its radii into six equilateral triangles,

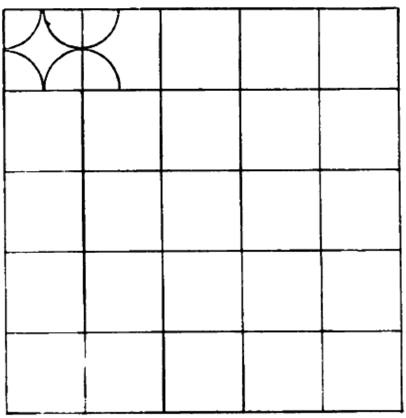

Fig. 15.

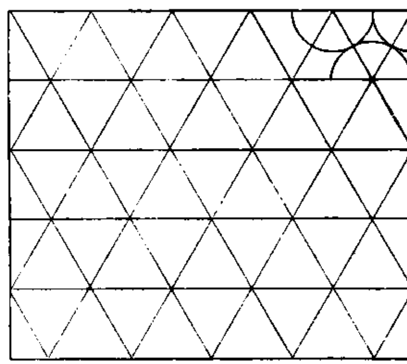

Fig. 16.

the division of the surface into triangles and hexagons gives the same arrangement (Fig. 16).

The hexagonal form of the bee's cell has long attracted attention and admiration. The little worker could not have chosen a better form if he had had the advantage of a full course in Euclid! The hexagon is best adapted to the purpose. It was discussed from a mathematical point of view by Maclaurin in one of the last papers he wrote.* It has been pointed out† that the hexagonal structure need not be attributed to mechanical instinct, but may be due solely to external pressure. (The

*In *Philosophical Transactions* for 1743.

†See for example E. P. Evans's *Evolutional Ethics and Animal Psychology*, p. 205.

cells of the human body, originally round, become hexagonal under pressure from morbid growth.)

Agricultural journals are advising the planting of trees (as also corn etc.) on the plan of the equilateral triangle instead of the square. Each tree is as far from its nearest neighbors in Fig. 16 as in Fig. 15. The circles indicated in the corner of each figure represent the soil etc. on which each tree may be supposed to draw. The circles in Fig. 16 are as large as in Fig. 15 but there is not so much space lost between them. As the distance from row to row in Fig. 16 = the altitude of one of the equilateral triangles $= \frac{1}{2}\sqrt{3} = 0.866$ of the distance between trees, it requires (beyond the first row) only 87% as much ground to set out a given number of trees on this plan as is required to set them out on the plan of Fig. 15. It may be predicted that, as land becomes scarce, *pressure* will force the orchards, gardens and fields into a uniformly hexagonal arrangement.

A HOMEMADE LEVELING DEVICE.

The newspapers have been printing instructions for making a simple instrument useful in laying out the grades for ditches on a farm, or in similar work in which a high degree of accuracy is not needed.

Strips of thin board are nailed together, as shown in Fig. 17, to form a triangle with equal vertical sides. The mid-point of the base is marked, and a plumb line is let fall from the opposite vertex. When the instrument is placed so that the line crosses the mark, the bar at the

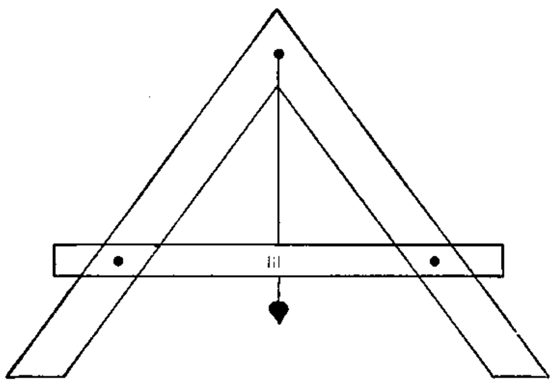

Fig. 17.

base is horizontal, being perpendicular to the plumb line. *The median to the base of an isosceles triangle is perpendicular to the base.* From the lengths of the sides of the triangle it may be computed—or it may be found by trial—how far from the middle of the crossbar a mark must be placed so that when the plumb line crosses it the bar shall indicate a grade of 1 in 200, 1 in 100, etc.

"ROPE STRETCHERS."

If a rope 12 units long be marked off into three segments of 3, 4, and 5 units, the end points brought together, and the rope stretched, the triangle thus formed is right-angled (Fig. 18). This was used by the builders of the pyramids. The Egyptian word for surveyor means,

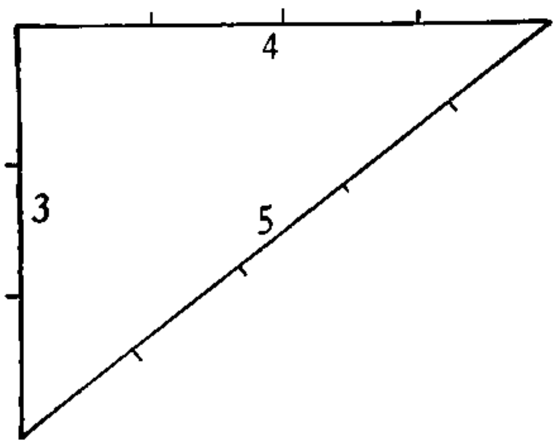

Fig. 18.

literally, "rope stretcher." Surveyors to this day use the same principle, counting off some multiple of these numbers in links of their chain.

THE THREE FAMOUS PROBLEMS OF ANTIQUITY.

1. To trisect an angle or arc.
2. To "duplicate the cube."
3. To "square the circle."

The trisection of an angle is an ancient problem; "but tradition has not enshrined its origin in romance."* The squaring of the circle is said to have been first attempted by Anaxagoras. The problem to duplicate the cube "was known in ancient times as the Delian problem, in consequence of a legend that the Delians had consulted Plato on the subject. In one form of the story, which is related by Philoponus, it is asserted that the Athenians in 430 B. C., when suffering from the plague of eruptive typhoid fever, consulted the oracle at Delos as to how they could stop it. Apollo replied that they must double the size of his altar which was in the form of a cube. To the unlearned suppliants nothing seemed more easy, and a new altar was constructed either having each of its edges double that of the old one (from which it followed that the volume was increased eightfold) or by placing a similar cubic altar next to the old one. Whereupon, according to the legend, the indignant god made the pestilence worse than before, and informed a fresh deputation that it was useless to trifle with him, as his new altar must be a cube and have a volume exactly double that of his old one. Suspecting a mystery the Athenians applied to Plato, who referred them to the geometricians, and especially to Euclid, who had made a special study of the problem."† It is a hard-hearted historical criticism that would cast a doubt on a story inherently so credible as this on account of so trifling a circumstance as that Plato was not born till 429 B. C. and Euclid much later.

Hippias of Elis invented the quadratrix for the trisection of an angle, and it was later used for the quadrature of the circle. Other Greeks devised other curves to effect the construction required in (1) and (2). Eratosthenes and Nicomedes invented mechanical instruments to draw

*Ball, *Recreations*, p. 245.

†Ball, *Hist.*, p. 43–4; nearly the same in his *Recreations*, p. 239–240.

such curves. But none of these curves can be constructed with ruler and compass alone. And this was the limitation imposed on the solution of the problems.

Antiquity bequeathed to modern times all three problems unsolved. Modern mathematics, with its more efficient methods, has proved them all impossible of construction with ruler and compass alone—a result which the shrewdest investigator in antiquity could have only conjectured—has shown new ways of solving them if the limitation of ruler and compass be removed, and has devised and applied methods of approximation. It has *dissolved* the problems, if that term may be permitted.

It was not until 1882 that the transcendental nature of the number π was established (by Lindemann). The final results in all three of the problems, with mathematical demonstrations, are given in Klein's *Famous Problems of Elementary Geometry.* A more popular and elementary discussion is Rupert's *Famous Geometrical Theorems and Problems.*

It should be noted that the number π, which the student first meets as the ratio of the circumference to the diameter of a circle, is a number that appears often in analysis in connections remote from elementary geometry; e. g., in formulas in the calculus of probability.

The value of π was computed to 707 places of decimals by William Shanks. His result (communicated in 1873) with a discussion of the formula he used (Machin's) may be found in the *Proceedings of the Royal Society of London*, Vol. 21. No other problem of the sort has been worked out to such a degree of accuracy—"an accuracy exceeding the ratio of microscopic to telescopic distances." An illustration calculated to give some conception of the degree of accuracy attained may be found in Professor Schubert's *Mathematical Essays and Recreations*, p. 140.

Shanks was a computer. He stands in contrast to the circle-squarers, who expect to find a "solution." Most of his computation serves, apparently, no useful purpose. But it should be a deterrent to those who—immune to the demonstration of Lindemann and others—still hope to find an exact ratio.

The quadrature of the circle has been the most fascinating of mathematical problems. The "army of circle-squarers" has been recruited in each generation. "Their efforts remained as futile as though they had attempted to jump into a rainbow" (Cajori); yet they were undismayed. In some minds, the proof that no solution can be found seems only to have lent zest to the search.

That these problems are of perennial interest, is attested by the fact that contributions to them still appear. In 1905 a little book was published in Los Angeles entitled *The Secret of the Circle and the Square*, in which also the division of "any angle into any number of equal angles" is considered. The author, J. C. Willmon, gives original methods of approximation. *School Science and Mathematics* for May 1906 contains a "solution" of the trisection problem by a high-school boy in Missouri, printed, apparently, to show that the problem still has fascination for the youthful mind. In a later number of that magazine the problem is discussed by another from the vantage ground of higher mathematics.

While the three problems have all been proved to be insolvable under the condition imposed, still the attempts made through many centuries to find a solution have led to much more valuable results, not only by quickening interest in mathematical questions, but especially by the many and important discoveries that have been made in the effort. The voyagers were unable to find the northwest passage, and one can easily see now that the search was *necessarily* futile; but in the attempt they discovered continents whose resources, when developed, make the wealth of the Indies seem poor indeed.

THE CIRCLE-SQUARER'S PARADOX.

Professor De Morgan, in his *Budget of Paradoxes* (London, 1872) gave circle-squarers the honor of more extended individual notice and more complete refutation than is often accorded them. The Budget first appeared in instalments in the *Athenæum*, where it attracted the correspondence and would-be contributions of all the circle-squarers, and the like amateurs, of the day. His facetious treatment of them drew forth their severest criticisms, which in turn gave most interesting material for the Budget. He says he means that the coming New Zealander shall know how the present generation regards circle-squarers. Theirs is one of the most amusing of the many paradoxes of which he wrote. The book is out of print, and so rare that the following quotations from it may be acceptable:

"Mere pitch-and-toss has given a more accurate approach to the quadrature of the circle than has been reached by some of my paradoxers. . . . The method is as follows: Suppose a planked floor of the usual kind, with thin visible seams between the planks. Let there be a thin straight rod, or wire, not so long as the breadth of the plank. This rod, being tossed up at hazard, will either fall quite clear of the seams, or will lay across one seam. Now Buffon, and after him Laplace, proved the following: That in the long run the fraction of the whole number of trials in which a seam is intersected will be the fraction which twice the length of the rod is of the circumference of the circle having the breadth of a plank for its diameter. In 1855 Mr. Ambrose Smith, of Aberdeen, made 3,204 trials with a rod three-fifths of the distance between the planks: there were 1,213 clear intersections, and 11 contacts on which it was difficult to decide. Divide these contacts equally . . . this gives $\pi = 3.1553$. A pupil of mine made 600 trials with a rod of the length between the seams, and got $\pi = 3.137$." (P. 170–1.)*

*Ball, in his *Mathematical Recreations and Essays* (p. 261, citing the *Messenger of Mathematics*, Cambridge, 1873, 2: 113–4) adds that "in 1864 Captain Fox made 1120 trials with some additional precautions, and obtained as the mean value $\pi = 3.1419$."

"The celebrated interminable fraction 3.14159 . . . , which the mathematician calls π, is the ratio of the circumference to the diameter. But it is thousands of things besides. It is constantly turning up in mathematics: and if arithmetic and algebra had been studied without geometry, π must have come in somehow, though at what stage or under what name must have depended upon the casualties of algebraical invention. This will readily be seen when it is stated that π is nothing but four times the series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$

ad infinitum. It would be wonderful if so simple a series had but one kind of occurrence. As it is, our trigonometry being founded on the circle, π first appears as the ratio stated. If, for instance, a deep study of probable fluctuation from the average had preceded geometry, π might have emerged as a number perfectly indispensable in such problems as— What is the chance of the number of aces lying between a million $+ x$ and a million $- x$, when six million of throws are made with a die?" (P. 171.)

"More than thirty years ago I had a friend . . . who was . . . thoroughly up in all that relates to mortality, life assurance, etc. One day, explaining to him how it should be ascertained what the chance is of the survivors of a large number of persons now alive lying between given limits of number at the end of a certain time, I came, of course, upon the introduction of π, which I could only describe as the ratio of the circumference of a circle to its diameter. 'Oh, my dear friend! that must be a delusion; what can the circle have to do with the numbers alive at the end of a given time?'—'I cannot demonstrate it to you; but it is demonstrated.' " (P. 172.)

"The feeling which tempts persons to this problem [exact quadrature] is that which, in romance, made it impossible for a knight to pass a castle which belonged to a giant or an enchanter. I once gave a lecture on the subject: a gentleman who was introduced to it by what I said remarked, loud enough to be heard all around, 'Only prove to me that it is impossible, and I will set about it this very evening.'

"This rinderpest of geometry cannot be cured, when once it has seated itself in the system: all that can be done is to apply what the learned call prophylactics to those who are yet sound." (P. 390.)

"The finding of two mean proportionals is the preliminary to the famous old problem of the duplication of the cube, proposed by Apollo (not Apollonius) himself. D'Israeli speaks of the 'six follies of science,'—the quadrature, the duplication, the perpetual motion, the philosopher's stone, magic, and astrology. He might as well have added the trisection, to make the mystic number seven: but had he done so, he would still have been very lenient; only seven follies in all science, from mathematics to chemistry! Science might have said to such a judge—as convicts used to say who got seven years, expecting it for life, 'Thank you, my Lord, and may you sit there till they are over,'—may the Curiosities of Literature outlive the Follies of Science!" (P. 71.)

THE INSTRUMENTS THAT ARE POSTULATED.

The use of two instruments is allowed in theoretic elementary geometry, the ruler and the compass—a limitation said to be due to Plato.

It is understood that the compass is to be of unlimited opening. For if the compass would not open as far as we please, it could not be used to effect the construction demanded in Euclid's third postulate, the drawing of a circle with any center and *any* radius. Similarly, it is understood that the ruler is of unlimited length for the use of the second postulate.

Also that the ruler is *ungraduated.* If there were even *two* marks on the straight-edge and we were allowed to use these and move the ruler *so as to fit* a figure, the problem to trisect an angle (impossible to elementary geometry) could be readily solved, as follows:

Let ABC be the angle, and P, Q the two points on the straight-edge. (Fig. 19.)

On one arm of angle B lay off BA = PQ. Bisect BA, at M.

Draw MK $\parallel$ BC, and ML $\perp$ BC.

Adjust the straight-edge to fit the figure so that P lies on MK, Q on ML, and at the same time the straight-edge passes through B. Then BP trisects the angle.

Proof. $\angle$PBC = its alternate $\angle$BPM.

Mark N the mid-point of PQ, and draw NM. Then N, the mid-point of the hypotenuse of the rt. $\triangle$ PQM, is equidistant from the vertexes of the triangle.

$$\therefore \angle \mathrm{BPM} = \angle \mathrm{PMN}.$$

$$\text{Exterior } \angle \mathrm{BNM} = \angle \mathrm{BPM} + \angle \mathrm{PMN}$$

$$= 2\angle \mathrm{BPM}.$$

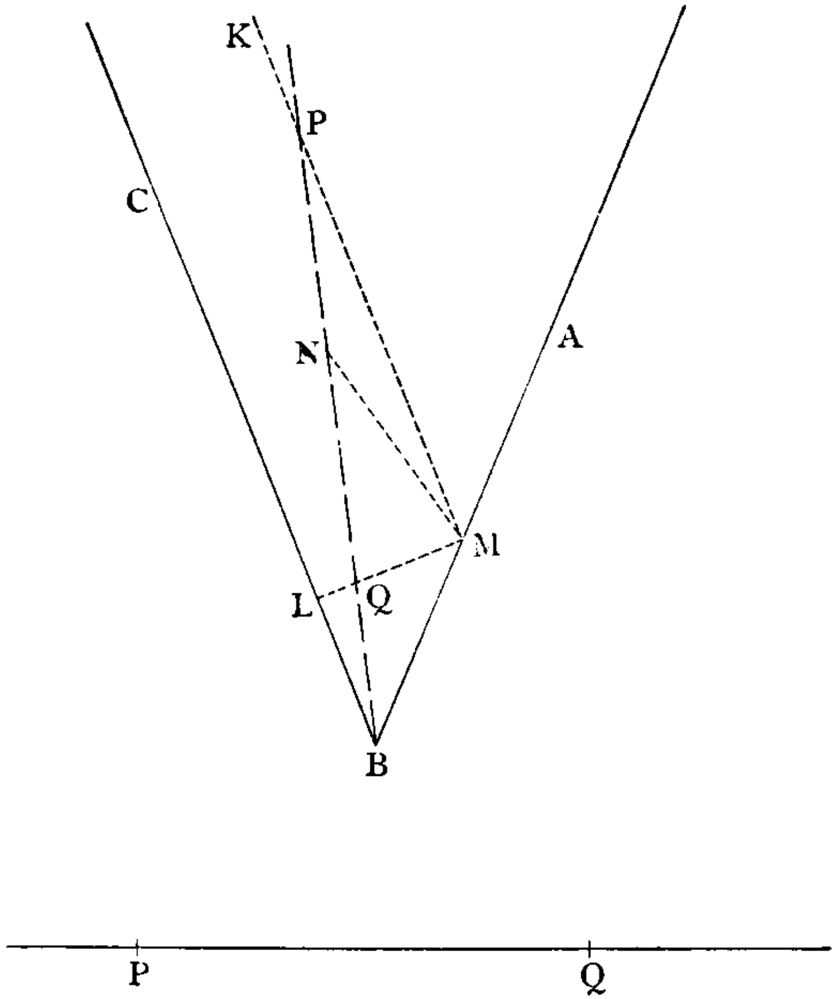

Fig. 19.

$$\because \text{NM} = \tfrac{1}{2}\text{PQ} = \text{BM}$$

$$\therefore \angle\text{MBN} = \angle\text{BNM}$$

$$\angle\text{PBC} = \angle\text{BPM} = \tfrac{1}{2}\angle\text{BNM} = \tfrac{1}{2}\angle\text{ABN} = \tfrac{1}{3}\angle\text{ABC}.$$

A. B. Kempe,* from whom this form of the well-known solution is adapted, raises the question whether Euclid does not use a graduated ruler and the fitting process when, in book 1, proposition 4, he fits side AB of triangle ΑΒΓ to side ΔE of triangle ΔEZ—the first proof by superposition, with which every high-school pupil is familiar. It may be replied that Euclid does not determine a point (as P is found in the angle above) by fitting and measuring. He superposes only in his reasoning, in his proof.

Our straight-edge must be ungraduated, or it grants us too much; it must be unlimited or it grants us too little.

* *How to Draw a Straight Line*, note (2).

THE TRIANGLE AND ITS CIRCLES.

The following statement of notation and familiar definitions may be permitted:

O, *orthocenter* of the triangle ABD, the point of concurrence of the three altitudes of the triangle.

G, center of *gravity*, center of mass, or centroid, of the triangle, the point of concurrence of the three medians.

C, *circumcenter* of the triangle, center of the circumscribed circle, point of concurrence of the perpendicular bisectors of the sides of the triangle.

I, *in-center* of the triangle, center of the inscribed circle, point of concurrence of the bisectors of the three interior angles of the triangle.

E, E, E, *ex-centers*, centers of the escribed circles, each E the point of concurrence of the bisectors of two exterior angles of the triangle and one interior angle.

An obtuse angled triangle is used in the figure so that the centers may be farther apart and the figure less crowded.

Collinearity of centers. O, G, and C are collinear, and OG = twice GC.

Corollary: The distance from O to a vertex of the triangle is twice the distance from C to the side opposite that vertex.*

The nine-point circle. Let L, M, N be the mid-points of the sides; A′, B′, D′, the projections of the vertexes on the opposite sides; H, J, K, the mid-points of OA, OB, OD, respectively. Then these nine points are concyclic; and the circle through them is called the nine-point circle of the triangle (Fig. 20).

The center of the nine-point circle is the mid-point of OC, and its radius is half the radius of the circumscribed circle.

*Or this corollary may easily be proved independently and the proposition that O, G, and C are in a straight line of which G is a trisection point be derived from it, as the writer once did when unacquainted with the results that had been achieved in this field.

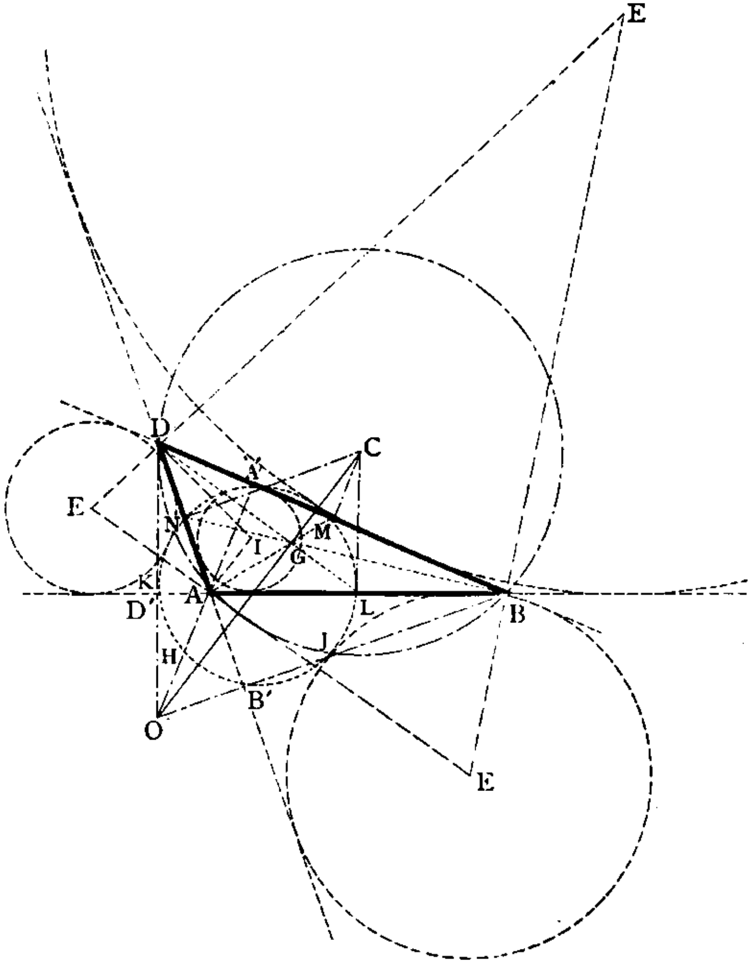

Fig. 20.

The discovery of the nine-point circle has been erroneously attributed to Euler. Several investigators discovered it independently in the early part of the nineteenth century. The name *nine-point circle* is said to be due to Terquem (1842) editor of *Nouvelles Annales*. Karl Wilhelm Feuerbach proved, in a pamphlet of 1822, what is now known as "Feuerbach's theorem": The nine-point circle of a triangle is tangent to the inscribed circle and each of the escribed circles of the triangle.

So many beautiful theorems about the triangle have been proved that Crelle—himself one of the foremost investigators of it—exclaimed: "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties. How many as yet unknown properties of other figures may there not be!"

The reader is referred to Cajori's *History of Elementary Mathematics* and the treatises on this subject mentioned in his note, p. 259, and to the delightful monograph, *Some Noteworthy Properties of the Triangle and Its Circles*, by W. H. Bruce, president of the North Texas State Normal School, Denton. Many of Dr. Bruce's proofs and some of his theorems are original.

Under the title *How to Draw a Straight Line*, A. B. Kempe wrote a little book which is full of theoretic interest to the geometer, as it touches one of the foundation postulates of the science.

We occasionally run a pencil around a coin to draw a circumference, thus using one circle to produce another. But this is only a makeshift: we have an instrument, not itself circular, with which to draw a circle—the compass. Now, when we come to draw a straight line we say that that postulate grants us the use of a ruler. But this is demanding a

Fig. 21.

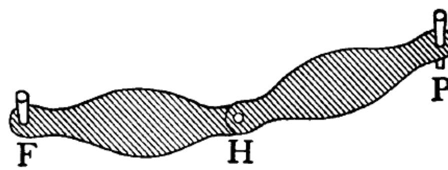

Fig. 22.

straight edge for drawing a straight line—given a straight line to copy. Is it possible to construct an instrument, not itself straight, which shall draw a straight line? Such an instrument was first invented by Peaucellier, a French army officer in the engineer corps. It is a "linkage." Since that time (1864) other linkages have been invented to effect rectilinear motion, some of them simpler than Peaucellier's. But as his is earliest, it may be taken as the type.

Preliminary to its construction, however, let us consider a single link (Fig. 21) pivoted at one end and carrying a pencil at the other. The pencil describes a circumference. If two links (Fig. 22) be hinged at H, and point F fastened to the plane, point P is free to move in any direction; its path is indeterminate. The number of links must be odd to give determinate motion. If a system of three links be fastened at both ends, a point in the middle link describes a definite curve—say a loop. Five links can give the requisite straight-line motion; but Peaucellier's was a seven-link apparatus.

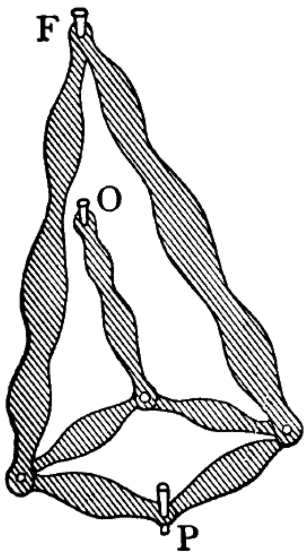

Fig. 23.

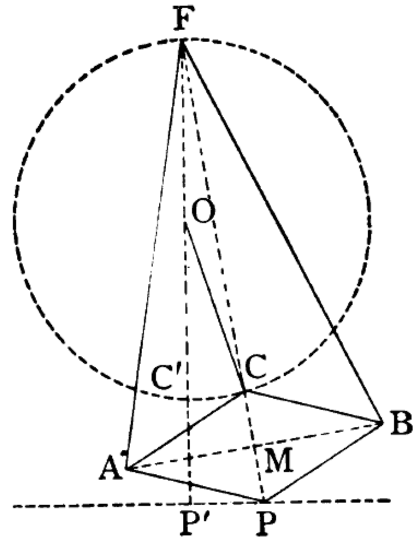

Fig. 24.

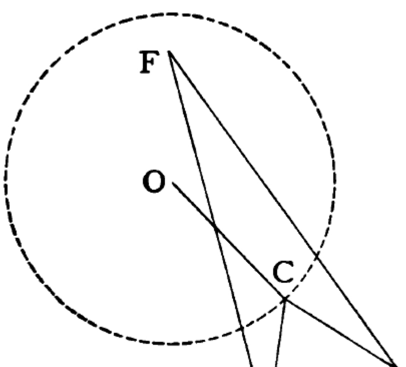

Fig. 25.

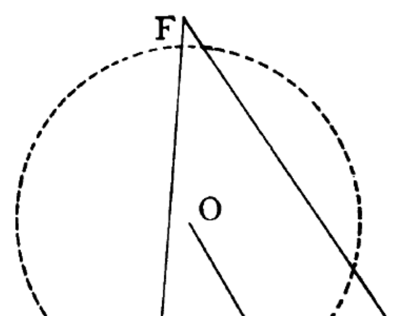

Fig. 26.

Such a linkage can be made by any teacher. The writer once made a small one of links cut out of cardboard and fastened together by shoemaker's eyelets; also a larger one (about 30 times the size of Fig. 23) of thin boards joined with bolts. F and O (Fig. 23) were made to fasten in mouldings above the blackboard, and P carried a piece of crayon. This proved very interesting to a geometry class for a lecture. It is needless to say that no one would think of any such appliance for daily class-room use. The ruler is the practical instrument.

Fig. 24 is a diagram of the apparatus shown in Fig. 23. FA = FB. In all positions APBC is a rhombus. F and O are fastened at points whose distance apart is equal to OC. Then C moves in an arc of a circle whose center is O; A and B move in an arc with center at F. It is to be shown that P moves in a straight line.

Draw $PP' \perp FO$ produced.* FCC′, being inscribed in a semicircle, is a right angle. Hence $\triangle$s FP′P and FC′C, having $\angle F$ in common, are similar, and

$$\begin{aligned} FP : FP' &= FC' : FC \\ FP \cdot FC &= FP' \cdot FC'. \qquad (1) \end{aligned}$$

F, C, and P, being each equidistant from A and B, lie in the same straight line; and the diagonals of the rhombus APBC are perpendicular bisectors of each other. Hence

$$\begin{aligned} FB^2 &= FM^2 + MB^2 \\ PB^2 &= MP^2 + MB^2 \\ \therefore FB^2 - PB^2 &= FM^2 - MP^2 \\ &= (FM + MP)(FM - MP) \\ &= FP \cdot FC \qquad (2) \end{aligned}$$

From (1) and (2), $FP' \cdot FC' = FB^2 - PB^2$.

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* *Imagine* these lines drawn, if one objects to drawing a straight line as one step in the process of showing that a straight line can be drawn!

But as the linkage moves, FC′, FB, and PB all remain constant; therefore FP′ is constant. That is, P′, the projection of P on FO, is always the same point; or in other words, P moves in a *straight line* (perpendicular to FO).

If the distance between the two fixed points, F and O, be made less than the length of the link OC, P moves in an arc of a circle with concave toward O (Fig. 25). As OC − OF approaches zero as a limit, the radius of the arc traced by P increases without limit.

Then as would be expected, if OF be made greater than OC, P traces an arc that is convex toward O (Fig. 26). The smaller OF − OC, the longer the radius of the arc traced by P. It is curious that so small an instrument may be used to describe an arc of a circle with enormous radius and with center on the opposite side of the arc from the instrument.

The straight line—the "simplest curve" of mathematicians—lies between these two arcs, and is the limiting form of each.

Linkages possess many interesting properties. The subject was first presented to English-speaking students by the late Professor Sylvester. Mr. Kempe showed "that a link-motion can be found to describe any given algebraic curve."

THE FOUR-COLORS THEOREM.

This theorem, known also as the map makers' proposition, has become celebrated. It is, that four colors are sufficient for any map, no two districts having a common boundary line to be colored the same; and this no matter how numerous the districts, how irregular their boundaries or how complicated their arrangement.

That four colors may be necessary can be seen from Fig. 27. A few trials will convince most persons that it is probably impossible to draw

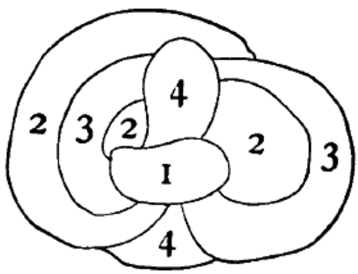

Fig. 27.

a map requiring more than four. To give a mathematical proof of it, is quite another matter.

The proposition is said to have been long known to map makers. It was mentioned as a mathematical proposition by A. F. Möbius, in 1840, and later popularized by De Morgan. All that is needed to give a proposition celebrity is to proclaim it one of the unsolved problems of the science. Cayley's remark, in 1878, that this one had remained unproved was followed by at least two published demonstrations within two or three years. But each had a flaw. The chance is still open for some one to invent a new method of attack.

If the proposition were not true, it could be disproved by a single special case, by producing a "map" with five districts of which each bounds every other. Many have tried to do this.

It has been shown that there are surfaces on which the proposition would not hold true. The theorem refers to a plane or the surface of a globe.

For historical presentation and bibliographic notes, see Ball's *Recreations*, pp. 51–3; or for a more extended discussion, Lucas, IV, 168 *et seq.*

PARALLELOGRAM OF FORCES.

One of the best-known principles of physics is, that if a ball, B, is struck a blow which if acting alone would drive the ball to A, and a blow which alone would drive it to C, and both blows are delivered at once, the ball takes the direction BD, the diagonal of the parallelogram

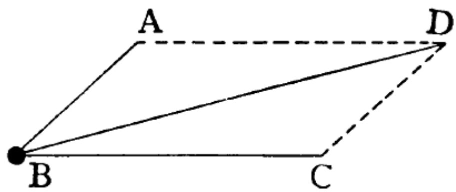

Fig. 28.

of BA and BC, and the force is just sufficient to drive the ball to D. BD is the *resultant* of the two forces.

If a third force, represented by some line BE, operates simultaneously with those represented by BA and BC, then the diagonal of the parallelogram of BD and BE is the resultant of the three forces. And so on.

Hence the resultant of forces is always less than the sum of the forces unless the forces act in the same direction. The more nearly their lines of action approach each other, the more nearly does their resultant approach their sum.

One is tempted to draw the moral, that social forces have a resultant and obey an analogous law, the result of all the educational or other social energy expended on a child, or in a community, being less than the sum, unless all forces act in the same line.

A QUESTION OF FOURTH DIMENSION BY ANALOGY.

After class one day a normal-school pupil asked the writer the following question, and received the following reply:

Q. If the path of a moving point (no dimension) is a line (one dimension), and the path of a moving line is a surface (two dimensions), and the path of a moving surface is a solid (three dimensions), why isn't the path of a moving solid a four-dimensional magnitude?

A. If your hypotheses were correct, your conclusion should follow by analogy. The path of a moving point is, indeed, always a line. The path of a moving line is a surface *except* when the line moves in its own dimension, "slides in its trace." The path of a moving surface is a solid only when the motion is in a third dimension. The generation of a four-dimensional magnitude by the motion of a solid presupposes that the solid is to be moved in a fourth dimension.

SYMMETRY ILLUSTRATED BY PAPER FOLDING.

The following simple device has been found by the writer to give pupils an idea of symmetry with a certainty and directness which no verbal explanation unaided can approach. Require each pupil to take a piece of calendered or sized paper, fold and crease it once, straighten it out again, draw rapidly with ink any figure on one half of the paper, and fold together while the ink is still damp. The original drawing and the trace on the other half of the paper are symmetric with respect to the crease as an axis. Again: Fold a paper in two perpendicular creases. In one quadrant draw a figure whose two end points lie one in each crease. Quickly fold so as to make a trace in each of the other quadrants. A closed figure is formed which is symmetric with respect to the intersection of the creases as center.

T. Sundara Row, in his *Geometric Exercises in Paper Folding* (edited and revised by Beman and Smith),* has shown how to make many of the constructions of plane geometry by paper folding, including beautiful illustrations of some of the regular polygons and the locating of points on some of the higher plane curves.

Illustrations of symmetry by the use of the mirror are well brought out in a brief article recently published in *American Education*.†

*Chicago, The Open Court Publishing Co.

†Number for March 1907, p. 464–5, article "Symmetrical Plane Figures," by Henry J. Lathrop.

APPARATUS TO ILLUSTRATE LINE VALUES OF TRIGONOMETRIC FUNCTIONS.

A piece of apparatus to illustrate trigonometric lines representing the trigonometric ratios may be constructed somewhat as follows (Fig. 29):

To the center O of a disc is attached a rod OR, which may be revolved. A tangent rod is screwed to the disc at A. Along this a little

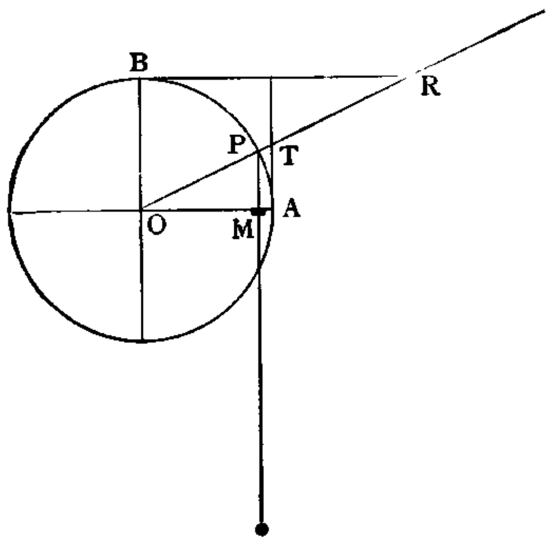

Fig. 29.

block bearing the letter T is made to slide easily. The block is also connected to the rod OR, so that T marks the intersection of the two lines. Similarly a block R is moved along the tangent rod BR. At P, a unit's distance from O on the rod OR, another rod (PM) is pivoted. A weight at the lower end keeps the rod in a vertical position. It passes through a block which is made to slide freely along OA and which bears the letter M.

As the rod OR is revolved in the positive direction, increasing the angle O, MP represents the increasing sine, OM the decreasing cosine, AT the increasing tangent, BR the decreasing cotangent, OT the increasing secant, OR the decreasing cosecant.

"SINE."

Students in trigonometry sometimes say: "From the line value, or geometric representation, of the trigonometric ratios it is easy to see why the tangent and secant were so named. And the co-functions are the functions of the complementary angles. But what is the origin of the name *sine*?" It is a good question. The following answer is that of Cantor, Fink, and Cajori; but Cantor deems it doubtful.

The Greeks used the entire chord of double the arc. The Hindus, though employing half the chord of double the arc (what we call *sine* in a unit circle), used for it their former name for the entire chord, *jîva*, which meant literally "bow-string," a natural designation for chord. Their work came to us through the Arabs, who transliterated the Sanskrit *jîva* into Arabic *dschiba*. Arabic being usually written in "unpointed text" (without vowels) like a modern stenographer's notes, *dschiba* having no meaning in Arabic, and the Arabic word *dschaib* having the same consonants, it was easy for the latter to take the place of the former. But *dschaib* means "bosom." Al Battani, the foremost astronomer of the ninth century, wrote a book on the motion of the heavenly bodies. In the twelfth century this was translated into Latin by Plato Tiburtinus, who rendered the Arabic word by the Latin *sinus* (bosom). And *sinus*, Anglicized, is "sine."

GROWTH OF THE PHILOSOPHY OF THE CALCULUS.

The latter half of the seventeenth century produced that powerful instrument of mathematical research, the differential calculus.* The master minds that invented it, Newton and Leibnitz, failed to clear the subject of philosophical difficulties.

Newton's reasoning is based on this initial theorem in the *Principia*: "Quantities, and the ratios of quantities, that during any finite time constantly approach each other, and before the end of that time approach nearer than any given difference, are ultimately equal." It is not surprising that neither this statement nor its demonstration gave universal satisfaction. The "zeros" whose ratio was considered in the method of fluxions were characterized by the astute Bishop Berkeley as "ghosts of departed quantities."

Leibnitz based his calculus on the principle that one may substitute for any magnitude another which differs from it only by a quantity infinitely small. This is assumed as "a sort of axiom." Pressed for an explanation, he said that, in comparison with finite quantities, he treated infinitely small quantities as *incomparables*, negligible "like grains of sand in comparison with the sea." This, if consistently held, should have made the calculus a mere method of approximation.

According to the explanations of both, strictly applied, the calcu-

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*Newton and Leibnitz invented it in the sense that they brought it to comparative perfection as an instrument of research. Like most epoch-making discoveries it had been foreshadowed. Cavalieri, Kepler, Fermat and many others had been working toward it. One must go a long way back into the history of mathematics to find a time when there was no suggestion of it. As this note is penned the newspapers bring a report that Mr. Hiberg, a Danish scientist, says he has recently discovered in a palimpsest in Constantinople, a hitherto unknown work on mathematics by Archimedes. "The manuscript, which is entitled 'On Method,' is dedicated to Eratosthenes, and relates to the applying of mechanics to the solution of certain problems in geometry. There is in this ancient Greek manuscript a method that bears a strong resemblance to the integral calculus of modern days, and is capable of being used for the solution of problems reserved for the genius of Leibnitz and Newton eighteen centuries later." (N. Y. Tribune.)

lus should have produced results that were close approximations. But instead, its results were absolutely accurate. Berkeley first, and afterward L. N. M. Carnot, pointed out that this was due to compensation of errors. This phase of the subject is perhaps nowhere treated in a more piquant style than in Bledsoe's *Philosophy of Mathematics*.

The method of limits permits a rigor of demonstration not possible to the pure infinitesimalists. Logically the methods of the latter are to be regarded as abridgments. As treated by the best writers the calculus is to-day on a sound philosophical basis. It is admirable for its logic as well as for its marvelous efficiency.

But many writers are so dominated by the thinking of the past that they still use the symbol 0 to mean sometimes "an infinitely small quantity" and sometimes absolute zero. Clearer thinking impels to the use of ι (iota) or i or some other symbol to mean an infinitesimal, denoting by 0 only zero.

This distinction implies that between their reciprocals. The symbol ∞, first used for an infinite by Wallis in the seventeenth century, has long been used both for a variable increasing without limit and for absolute infinity. The revised edition of Taylor's Calculus (Ginn 1898) introduced a new symbol $\alpha\!\varphi$, a contraction of $a/0$, for absolute infinity, using ∞ only for an infinite (the reciprocal of an infinitesimal). It is to be hoped that this usage will become universal.

In the book just referred to is perhaps the clearest and most concise statement to be found anywhere of the inverse problems of the differential calculus and the integral calculus, as well as of the three methods used in the calculus.

SOME ILLUSTRATIONS OF LIMITS.

Physical illustrations of variables are numerous. But to find a similar case of a constant, is not easy. The long history of the determination of standards (yard, meter etc.) is the history of a search for physical constants. Constants are the result of abstraction or are limited by definition. Non-physical constants are numerous, and enter into most problems.

If one person is just a year older than another, the ratio of the age of the younger to that of the older, at successive birthdays, is $\frac{0}{1}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5} \cdots \frac{49}{50}, \frac{50}{51} \ldots$. In general: the ratio of the ages of any two persons is a variable approaching unity as limit. The sum of of their ages is a variable increasing without limit. The difference between their ages is a constant.

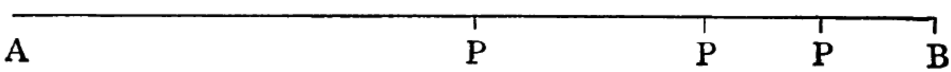

Fig. 30.

When pupils have the idea of the time-honored point P which moves half way from A to B the first second, half the remaining distance the next second, etc., but have trouble with the product of a constant and a variable, they have sometimes been helped by the following "optical illustration": Imagine yourself looking at Fig. 30 through a glass that makes everything look twice as large as it appears to the naked eye. AP still seems to approach AB as limit; that is, twice the "real" AP is approaching twice the "real" AB as limit. Now suppose your glass magnifies 3 times, n times. AP still approaches AB magnified the same number of times. That is, if AP $\doteq$ AB, then any constant $\times$ AP $\doteq$ that constant $\times$ AB.

Reverse the glass, making AP look one-nth part as large as at first. It approaches one-nth of the "real" AB. Putting this in symbols, with

x representing the variable, and c the constant,

$$\lim\left(\frac{x}{c}\right) = \frac{\lim x}{c}.$$

Or in words: The limit of the ratio of a variable to a constant is the ratio of the limit of the variable to the constant.

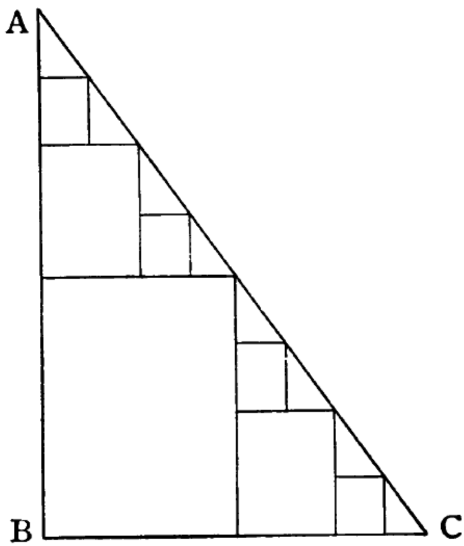

Fig. 31.

Let x represent the broken line from A to C (Fig. 31), composed first of 4 parts, then of 8, then of 16 (the last division shown in the figure) then of 32, etc. The polygon bounded by x, AB and BC $\doteq \triangle$ABC. What of the length of x? Most persons to whom this old figure is new answer off-hand, "$x \doteq$ AC." But a minute's reflection shows that x is constant and $=$ AB $+$ BC.

LAW OF COMMUTATION.

This law, emphasized for arithmetic in McLellan and Dewey's *Psychology of Number*, and explicitly employed in all algebras that give attention to the logical side of the subject, is one whose importance is often overlooked. So long as it is used implicitly and regarded as of universal application, its import is neglected. An antidote: to remember that there are regions in which this law does not apply. E. g.:

In the "geometric multiplication" of rectangular vectors used in quaternions, the commutative property of factors does not hold, but a change in the cyclic order of factors reverses the sign of the product.

Even in elementary algebra or arithmetic, the commutative principle is not valid in the operation of involution. Professor Schubert, in his *Mathematical Essays and Recreations*, has called attention to the fact that this limitation—the impossibility of interchanging base and exponent—renders useless any high operation of continued involution.

EQUATIONS OF U.S. STANDARDS OF LENGTH AND MASS.

By order approved by the secretary of the treasury April 5, 1893, the international prototype meter and kilogram are regarded as fundamental standards, the yard, pound etc. being defined in terms of them.

All of the nations taking part in the convention have very accurate copies of the international standards. The degree of accuracy of the comparisons may be seen from the equations expressing the relation of meter no. 27 and kilogram no. 20, of the United States, to the international prototypes. T represents the number of degrees of the centigrade scale of the hydrogen thermometer. The last term in each equation shows the range of error.

$$\text{M no. } 27 = 1\,\text{m} - 1.6\mu + 8.657\mu\text{T} + 0.00100\mu\text{T}^2 \pm 0.2\mu$$
$$\text{K no. } 20 = 1\,\text{kg} - 0.039\,\text{mg} \pm 0.002\,\text{mg}.$$

(U.S. coast and geodetic survey.)

THE MATHEMATICAL TREATMENT OF STATISTICS.

This is one of the most important and interesting applications of mathematics to the needs of modern civilization. Just as data gathered by an incompetent observer are worthless—or by a biased observer, unless the bias can be measured and eliminated from the result—so also conclusions obtained from even the best data by one unacquainted with the principles of statistics must be of doubtful value.

The laws of statistics are applications of mathematical formulas, especially of permutations, combinations and probability. Take for illustration two simple laws (the mathematical derivation of them would not be so simple):

1. Suppose a number of measurements have been made. If the measures be laid off as abscissas, and the number of times each measure occurs be represented graphically as the corresponding ordinate, the line drawn through the points thus plotted is called the *distribution curve* for these measures. The area between this line and the axis of x is the *surface of frequency*.

If a quantity one is measuring is due to chance combinations of an infinite number of causes, equal in amount and independent, and all equally likely to occur, the surface of frequency is of the form shown in Fig. 32, the equation of the curve being $y = e^{-x^2}$.

Most effects that are measured are not due to such combinations of causes, and their distribution curves are more or less irregular; but under favorable conditions they frequently approximate this, which may be called the normal, "the normal probability integral." In these cases the tables that have been computed for this surface are of great assistance.

2. Every one knows that, other things being equal, the greater the number of measurements made, the greater the probability of their average (or other mean) being the true one. It is shown mathematically that the probability *varies as the square root* of the number of measures. E. g., if in one investigation 64 cases were measured, and in another 25 cases, the returns from the first investigation will be more

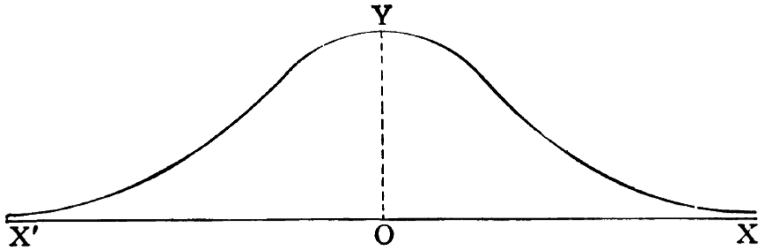

Fig. 32.

trustworthy than those from the second in the ratio of 8 to 5.

It is also apparent that, if the average deviation (or other measure of variability) of the measures from their average in one set is greater than in another, the average is less trustworthy in that set in which the variability is the greater. Expressed mathematically, the trustworthiness *varies inversely* as the variability. E. g., in one investigation the average deviation of the measures from their average is 2 (2 cm, 2 grams, or whatever the unit may be) while in another investigation (involving the same number of measures etc.) the average deviation is 2.5. Then the probable approach to accuracy of the average obtained in the first investigation is to that of the average obtained in the second as 1/2 is to 1/2.5, or as 5 to 4.

If the two investigations differed both in the number of measures and in the deviation from the average, both would enter as factors in determining the relative confidence to be reposed in the two results. E. g., combine the examples in the two preceding paragraphs: An average was obtained from 64 measures whose variability was 2, and another from 25 measures whose variability was 2.5. Then

$$\left.\begin{matrix}\text{trustworthiness}\\ \text{of first average}\end{matrix}\right\} : \left.\begin{matrix}\text{trustworthiness}\\ \text{of second average}\end{matrix}\right\} = \sqrt{64} \times \frac{1}{2} : \sqrt{25} \times \frac{1}{2.5}$$
$$= 2 : 1.$$

The trustworthiness of the mean of a number of measures varies directly as the square root of the number of measures and inversely as their variability.

The foregoing principles—the A B C of statistical science—show some of its method and its value and the direction in which it is working. Perhaps the most readable treatise on the subject is Professor Edward L. Thorndike's *Introduction to the Theory of Mental and Social Measurements*. It presupposes only an elementary knowledge of mathematics and contains references to more technical works on the subject.

Professor W. S. Hall, in "Evaluation of Anthropometric Data" (*Jour. Am. Med. Assn.*, Chicago, 1901) showed that the curve of distribution of biologic data is the curve of the coefficients in the expansion of an algebraic binomial. In a most interesting article, "A Guide to the Equitable Grading of Students," in *School Science and Mathematics* for June, 1906, he applies this principle to the distribution of student records in a class.

In the expansion of $(a+b)^5$ there are 6 terms, and the coefficients are 1, 5, 10, 10, 5, 1. Their sum is 32. If 320 students do their work and are tested and graded under normal (though perhaps unusual) conditions, and 6 different marks are used—say A, B, C, D, E, F—the number of pupils attaining each of these standings should approximate 10, 50, 100, 100, 50, 10, respectively. If 3200 students were rated in 6 groups under similar conditions, the numbers in the groups would be ten times as great—100, 500, 1000 etc., and the approximation would be relatively closer than when only 320 were tested. The study of the conditions that cause deviation from this normal distribution of standings is instructive both statistically and pedagogically.

A rough-and-ready statistical method, available in certain cases, may be illustrated as follows: Suppose we are engaged in ascertaining the number of words in the vocabulary of normal-school juniors. (Such an investigation is now in progress under the direction of Dr. Margaret K. Smith, of the normal faculty at New Paltz.) Let us select at random a page of the dictionary—say the 13th—and by appropriate tests ascertain the number of words on this page that the pupil knows, divide this number by the number of words on the page, and thus obtain a convenient expression for the *part* of the words known. Suppose the quotient to be .3016. Turn to page 113 and make similar tests, and divide the number known on *both* pages by the number of words on both pages, giving—say—.2391. After trying page 213 the result is found for three pages. In each case the decimal represents the total result reached thus far in the experiment. Suppose the successive decimals to be

.3016
.2391
.2742
.2688
.2562
.2610
.2628
.2631
.2642
.2638

A few decimals thus obtained may convince the experimenter that the first figure has "become constant." Many more may be necessary to determine the second figure unless the "series converges" rapidly as above. If the first two figures be found to be 26, this student knows 26% of the words in the dictionary. Multiplying the "dictionary total" by this coefficient, gives the extent of the student's vocabulary, correct to 1% of the dictionary total. If a higher degree of accuracy had been required, a three-place coefficient would have been determined.

This method has the practical advantage that the coefficient found at each step furnishes, by comparison with the coefficients previously obtained, an indication of the degree of accuracy that will be attained by its use. The labor of division may be diminished by using, on each page, only the first 20 words (or other multiple of 10). Similarly with each student to be examined. The method here described is applicable to certain classes of measures.

MATHEMATICAL SYMBOLS.

The origin of most of the symbols in common use may be learned from any history of mathematics. The noteworthy thing is their recentness. Of our symbols of operation the oldest are $+$ and $-$, which appear in Widmann's arithmetic (Leipsic, 1489).

Consider the situation in respect to symbols at the middle of the sixteenth century. The radical sign had been used by Rudolff, ( ), $\times$, $\div$, $>$, and $<$ were still many years in the future, $=$ had not yet appeared (though another symbol for the same had been used slightly) and $+$ and $-$ were not in general use. Almost everything was expressed by words or by mere abbreviations. Yet at that time both cubic and biquadratic equations had been solved and the methods published. It is astonishing that men with the intellectual acumen necessary to invent a solution of equations of the third or fourth degree should not have hit upon a device so simple as symbols of operation for the abridgment of their work.

The inconvenience of the lack of symbols may be easily tested by writing—say—a quadratic equation and solving it without any of the ordinary symbols of algebra.

Even after the introduction of symbols began, the process was slow. But recently it has moved with accelerating velocity, until now not only do we have a symbol for each operation—sometimes a choice of symbols—but most of the letters of the alphabet are engaged for special mathematical uses. E. g.:

a finite quantity, known number, side of triangle opposite A, intercept on axis of x, altitude ...

b known number, side of triangle opposite B, base, intercept on axis of y ...

c constant ...

d differential, distance ...

e base of Napierian logarithms.

A considerable inroad has been made on the Greek alphabet, e. g.:

γ inclination to axis of x.

π 3.14159...

Σ sum of terms similarly obtained.

σ standard deviation (in theory of measurements).

But the supply of alphabets is by no means exhausted. There is no cause for alarm.

BEGINNINGS OF MATHEMATICS ON THE NILE.

Whatever the excavations in Babylonia and Assyria may ultimately reveal as to the state of mathematical learning in those early civilizations, it is established that in Egypt the knowledge of certain mathematical facts and processes was so ancient as to have left no record of its origin.

The truth of the Pythagorean theorem for the special case of the isosceles right triangle may have been widely known among people using tile floors (see Beman and Smith's *New Plane Geometry*, p. 103). That 3, 4, and 5 are the sides of a right triangle was known and used by the builders of the pyramids and temples. The Ahmes papyrus (1700 B. C. and based on a work of perhaps 3000 B. C. or earlier) contains many arithmetical problems, a table of unit-fractions, etc., and the solution of simple equations, in which *hau* (heap) represents the unknown. Though one may feel sure that arithmetic must be the oldest member of the mathematical family, still the beginnings of arithmetic, algebra and geometry are all prehistoric. When the curtain raises on the drama of human history, we see men computing, solving linear equations, and using a simple case of the Pythagorean proposition.

A FEW SURPRISING FACTS IN THE HISTORY OF MATHEMATICS.

That spherical trigonometry was developed earlier than plane trigonometry (explained by the fact that the former was used in astronomy).

That the solution of equations of the third and fourth degree preceded the use of most of the symbols of operation, even of =.

That decimals—so simple and convenient—should not have been invented till after so much "had been attempted in physical research and numbers had been so deeply pondered" (Mark Napier).

That logarithms were invented before exponents were used; the derivation of logarithms from exponents—now always used in teaching logarithms—being first pointed out by Euler more than a century later.

That the earliest systems of logarithms (Napier's, Speidell's), constructed for the sole object of facilitating computation, should have missed that mark (leaving it for Briggs, Gellibrand, Vlacq, Gunter and others) but should have attained theoretical importance, lending themselves to the purposes of modern analytical methods (Cajori).

QUOTATIONS ON MATHEMATICS.

Following are some of the quotations that have been used at different times in the decoration of a frieze above the blackboard in the writer's recitation room:

Let no one who is unacquainted with geometry LEAVE here. (This near the door and on the inside—an adaptation of the motto that Plato is said to have had over the outside of the entrance to his school of philosophy, the Academy: "Let no one who is unacquainted with geometry *enter* here.")

God geometrizes continually. *Plato.*

There is no royal road to geometry. *Euclid.*

Mathematics, the queen of the sciences. *Gauss.*

Mathematics is the glory of the human mind. *Leibnitz.*

Mathematics is the most marvelous instrument created by the genius of man for the discovery of truth. *Laisant.*

Mathematics is the indispensable instrument of all physical research. *Berthelot.*

All my physics is nothing else than geometry. *Descartes.*

There is nothing so prolific in utilities as abstractions. *Faraday.*

The two eyes of exact science are mathematics and logic. *De Morgan.*

All scientific education which does not commence with mathematics is, of necessity, defective at its foundation. *Compte.*

It is in mathematics we ought to learn the general method always followed by the human mind in its positive researches. *Compte.*

A natural science is a science only in so far as it is mathematical. *Kant.*

The progress, the improvement of mathematics are linked to the prosperity of the state. *Napoleon.*

If the Greeks had not cultivated conic sections, Kepler could not have superseded Ptolemy. *Whewell.*

No subject loses more than mathematics by any attempt to dissociate it from its history. *Glaisher.*

AUTOGRAPHS OF MATHEMATICIANS.

For the photograph from which this cut (Fig. 33) was made the writer is indebted to Prof. David Eugene Smith. As an explorer in the bypaths of mathematical history and a collector of interesting specimens therefrom, Dr. Smith is, perhaps, without a peer.*

The reader will be interested to see a facsimile of the handwriting of Euler and Johann Bernoulli, Lagrange and Laplace and Legendre, Clifford and Dodgson, and William Rowan Hamilton, and others of the immortals, grouped together on one page. In the upper right corner is the autograph of Moritz Cantor, the historian of mathematics. On the sheet overlapping that, the name over the verses is faint; it is that of J. J. Sylvester, late professor in Johns Hopkins University.

One who tries to decipher some of these documents may feel that he is indeed "In the Mazes of Mathematics."† Mathematicians are not as a class noted for the elegance or the legibility of their chirography, and these examples are not submitted as models of penmanship. But each bears the sign manual of one of the builders of the proud structure of modern mathematics.

*Several handsome sets of portraits of mathematicians, edited by Dr. Smith, are published by The Open Court Publishing Company.

†This section first printed in a series bearing that title, in *The Open Court*, March–July, 1907.

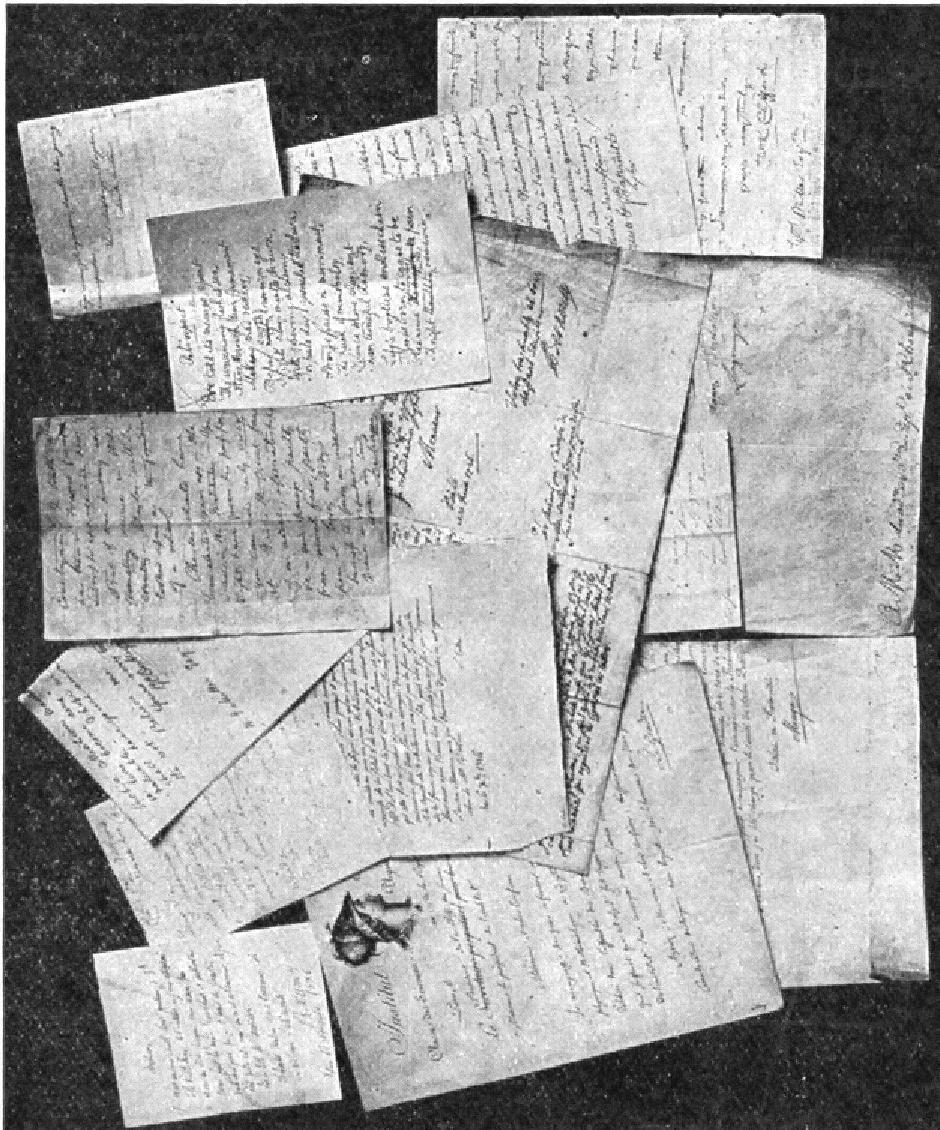

Fig. 33.

BRIDGES AND ISLES, FIGURE TRACING, UNICURSAL SIGNATURES, LABYRINTHS.

This section presents a few of the more elementary results of the application of mathematical methods to these interesting puzzle questions.*

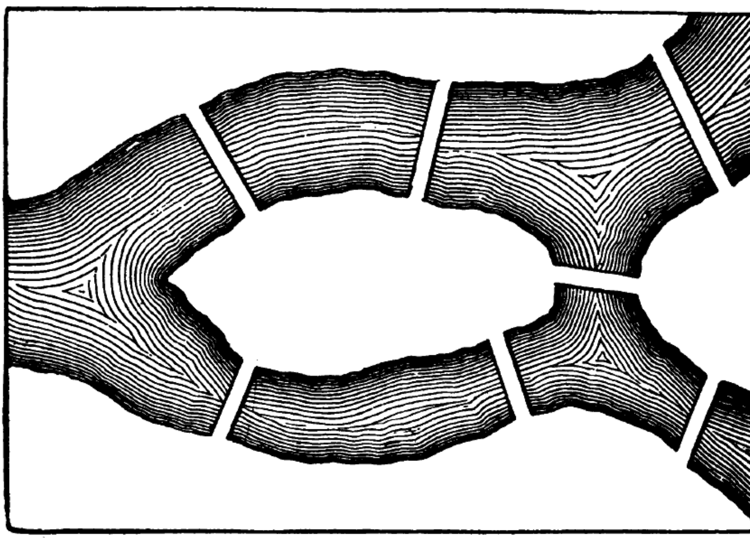

Fig. 34.

The city of Königsberg is near the mouth of the Pregel river, which has at that point an island called Kneiphof. The situation of the seven bridges is shown in Fig. 34. A discussion arose as to whether it is possible to cross all the bridges in a single promenade without crossing any bridge a second time. Euler's famous memoir was presented to

*For more extended discussion, and for proofs of the theorems here stated, see Euler's *Solutio Problematis ad Geometriam Situs Pertinentis*, Listing's *Vorstudien sur Topologie*, Ball's *Mathematical Recreations and Essays*, Lucas's *Récréations Mathématiques*, and the references given in notes by the last two writers named. To these two the present writer is especially indebted.

the Academy of Sciences of St. Petersburg in 1736 in answer to this question. Rather, the Königsberg problem furnished him the occasion to solve the general problem of any number and combination of isles and bridges.

Conceive the isles to shrink to points, and the problem may be stated more conveniently with reference to a diagram as the problem of tracing a given figure without removing the pencil from the paper

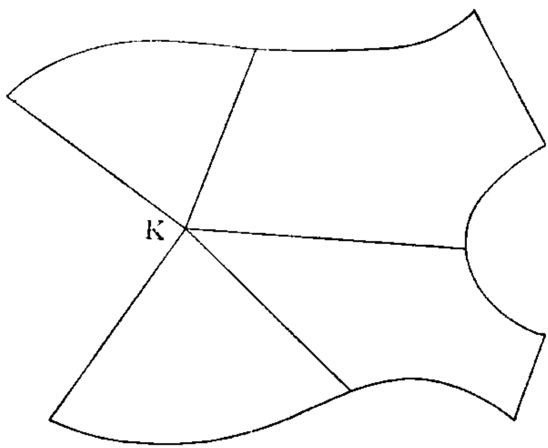

Fig. 35.

and without retracing any part; or, if not possible to do so with one stroke, to determine *how many* such strokes are necessary. Fig. 35 is a diagrammatic representation of Fig. 34, the isle Kneiphof being at the point K.

The number of lines proceeding from any point of a figure may be called the *order* of that point. Every point will therefore be of either an even order or an odd order. E. g., as there are 3 lines from point A of Fig. 36, the order of the point is odd; the order of point E is even. The well-known conclusions reached by Euler may now be stated as follows:

In a closed figure (one with no free point or "loose end") *the number*

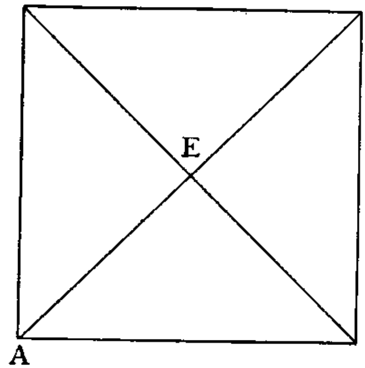

Fig. 36.

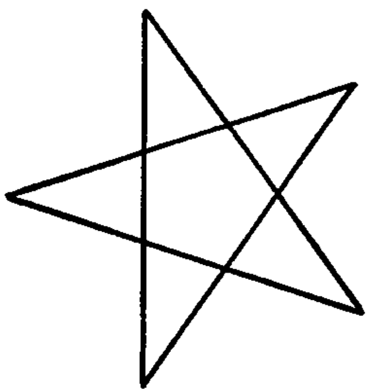

Fig. 37.

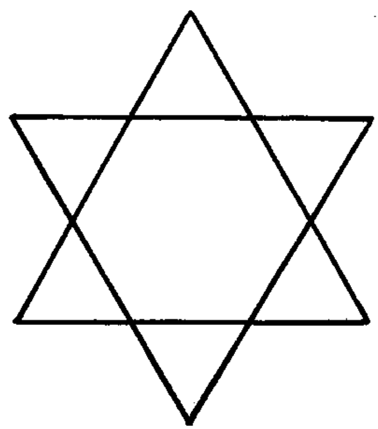

Fig. 38.

of points of odd order is even, whether the figure is unicursal or not. E. g., Fig. 36, a multicursal closed figure, has *four* points of odd order.

A figure of which every point is of even order can be traced by one stroke starting from any point of the figure. E. g., Fig. 37, the magic pentagon, symbol of the Pythagorean school, and Fig. 38, a "magic hexagram commonly called the shield of David and frequently used on synagogues" (Carus), have no points of odd order; each is therefore unicursal.

A figure with only two points of odd order can be traced by one stroke by starting at one of those points. E. g., Fig. 39 (taken originally from Listing's *Topologie*) has but two points of odd order, A and Z; it may therefore be traced by one stroke beginning at either of these two points and ending at the other. One may make a game of it by

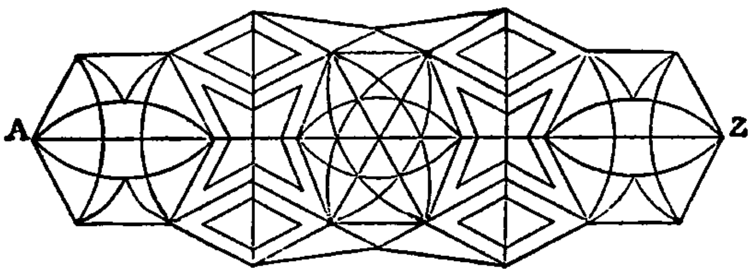

Fig. 39.

drawing a figure, as Lucas suggests, like Fig. 39, but in larger scale on cardboard, placing a small counter on the middle of each line that joins two neighboring points, and setting the problem to determine the course to follow in removing all the counters successively (simply tracing continuously and removing each counter as it is passed, an objective method of recording which lines have been traced).

A figure with more than two points of odd order is multicursal. E. g., Fig. 40 has more than two points of odd order and requires more than one course, or stroke, to traverse it.

The last two theorems just stated are special cases of Listing's:

Let $2n$ *represent the number of points of odd order; then* n *strokes are necessary and sufficient to trace the figure.* E. g., Fig. 39 with 2 points of odd order, requires one stroke; Fig. 40, representing a fragment of masonry, has 8 points of odd order and requires four strokes.

Return now to the Königsberg problem of Fig. 34. By reference to the diagram in Fig. 35 it is seen that there are four points of odd order.

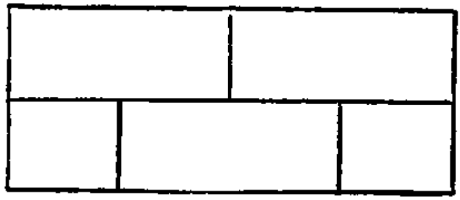

Fig. 40.

Hence it is not possible to cross every bridge once and but once without taking two strolls.

An interesting application of these theorems is the consideration of the number of strokes necessary to describe an n-gon and its diagonals. As the points of intersection of the diagonals are all of even order, we need to consider only the vertexes. Since from each vertex there is a line to every other vertex, the number of lines from each vertex is $n-1$. Hence, if n is odd, every point is of even order, and the entire figure can be traced unicursally beginning at any point; e. g., Fig. 41, a pentagon

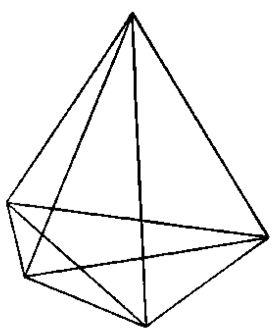

Fig. 41.

with its diagonals. If n is even, $n - 1$ is odd, every vertex is of odd order, the number of points of odd order is n, and the figure can not be described in less than $n/2$ courses; e.g., Fig. 36, quadrilateral, requires two strokes.

Unicursal signatures. A signature (or other writing) is of course subject to the same laws as are other figures with respect to the number of times the pen must be put to the paper. Since the terminal point could have been connected with the point of starting without lifting the pen, the signature may be counted as a closed figure if it has no free end but these two. The number of points of odd order will be found to be even. The dot over an *i*, the cross of a *t*, or any other

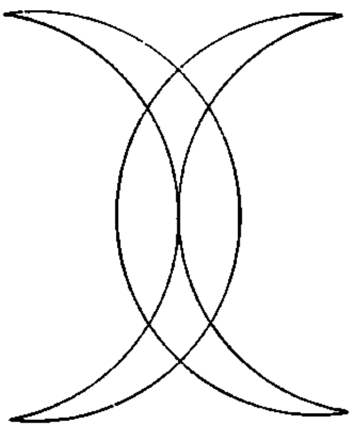

Fig. 42.

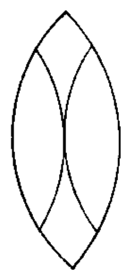

Fig. 43.

mark leaving a free point, makes the signature multicursal. There are so many names not requiring separate strokes that one would expect more unicursal signatures than are actually found. De Morgan's (as shown in the cut in the preceding section) is one; but most of the signatures there shown were made with several strokes each. Of the signatures to the Declaration of Independence there is not one that is strictly unicursal; though that of *Th Jefferson* looks as if the end of the *h* and the beginning of the *J* might often have been completely joined, and in that case his signature would have been written in a single course of the pen.

Fig. 42, formed of two crescents, is "the so-called sign manual of Mohammed, said to have been originally traced in the sand by the point of his scimetar without taking the scimetar off the ground or retracing any part of the figure," which can easily be done beginning at any point of the figure, as it contains no point of odd order. The mother of the writer suggests that, if the horns of Mohammed's crescents be

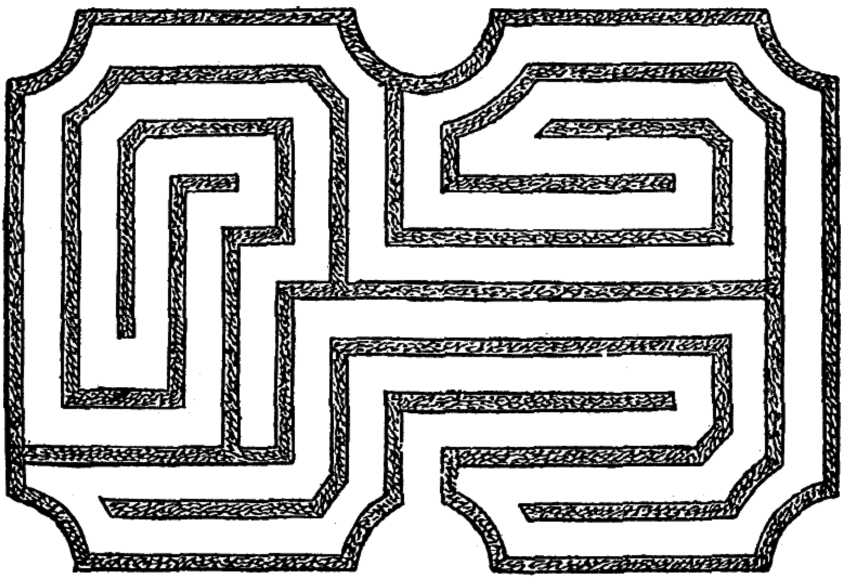

Fig. 44.

omitted, a figure (Fig. 43) is left which can not be traced unicursally. There are then four points of odd order; hence two strokes are requisite to describe the figure.

Labyrinths such as the very simple one shown in Fig. 44 (published in 1706 by London and Wise) are familiar, as drawings, to every one. In some of the more complicated mazes it is not so easy to thread one's way, even in the drawing, where the entire maze is in sight, while in the actual labyrinth, where walls or hedges conceal everything but the path one is taking at the moment, the difficulty is greatly increased and one needs a rule of procedure.

The mathematical principles involved are the same as for tracing other figures; but in their application several differences are to be noticed in the conditions of the two problems. A labyrinth, as it stands, is not a closed figure; for the entrance and the center are free ends, as are also the ends of any blind alleys that the maze may contain. These are therefore points of odd order. There are usually other points of odd order. Hence in a single trip the maze can not be completely traversed. But it is not required to do so. The problem here is, to go from the entrance to the center, the shorter the route found the better. Moreover, the rules of the game do not forbid retracing one's course.

It is readily seen (as first suggested by Euler) that by going over each line twice the maze becomes a closed figure, terminating where it begins, at the entrance, including the center as one point in the course, and containing only points of even order. Hence every labyrinth can be completely traversed by going over every path twice—once in each direction. It is only necessary to have some means of marking the routes already taken (and their direction) to avoid the possibility of losing one's way. This duplication of the entire course permits no failure and is so general a method that one does not need to know anything about the particular labyrinth in order to traverse it successfully and confidently. But if a plan of the labyrinth can be had, a course may be found that is shorter.

Theseus, as he *threaded* the Cretan labyrinth in quest of the Minotaur, would have regarded Euler's mathematical theory of mazes as much less romantic than the silken cord with Ariadne at the outer end; but there are occasions where a modern finds it necessary to "go by the book." Doubtless the labyrinth of Daedalus was "a mighty maze, but not without a plan."

Fig. 45 presents one of the most famous labyrinths, though by no means among the most puzzling. It is described in the *Encyclopædia Britannica* (article "Labyrinth") as follows:

"The maze in the gardens at Hampton Court Palace is considered to be one of the finest examples in England. It was planted in the early

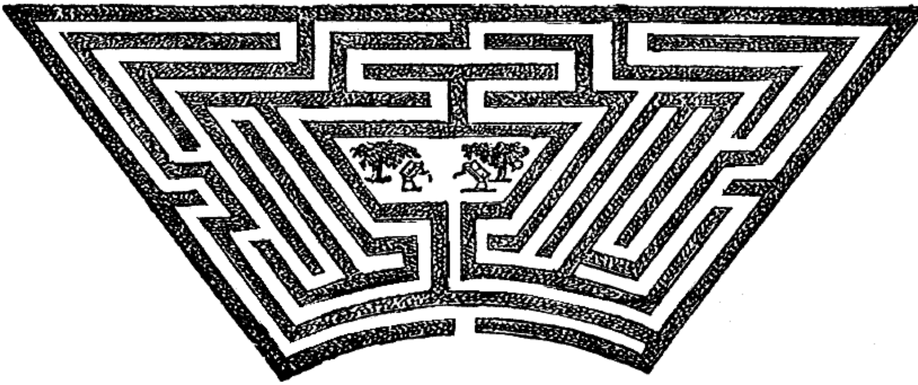

Fig. 45.

part of the reign of William III, though it has been supposed that a maze had existed there since the time of Henry VIII. It is constructed on the hedge and alley system, and was, we believe, originally planted with hornbeam, but many of the plants have died out, and been replaced by hollies, yews, etc., so that the vegetation is mixed. The walks are about half a mile in length, and the extent of ground occupied is a little over a quarter of an acre. The center contains two large trees, with a seat beneath each. The key to reach this resting place is to keep the right hand continuously in contact with the hedge from first to last, going round all the stops."

THE NUMBER OF THE BEAST.

"Here is wisdom. He that hath understanding, let him count the number of the beast; for it is the number of a man: and his number is Six hundred and sixty and six." (Margin, "Some ancient authorities read *Six hundred and sixteen*.") Revelation 13:18.

No wonder that these words have been a powerful incentive to a class of interpreters who delight in apocalyptic literature, especially to such as have a Pythagorean regard for hidden meaning in numbers.

There were centuries in which no satisfactory interpretation was generally known. At about the same time, in 1835, Benary, Fritzsche, Hitzig and Reuss connected the number 666 with "Emperor (Cæsar) Neron," [Hebrew]. In the number notation of the Hebrews the letter [H*ebrew*] = 100, [H*ebrew*] = 60, [H*ebrew*] = 200, [H*ebrew*] = 50, [H*ebrew*] = 200, [H*ebrew*] = 6, [H*ebrew*] = 50. These numbers added give 666. Omitting the final letter from the name (making it "Emperor Nero") the number represented is 616, the marginal reading. The present writer's casual opinion is that the foregoing is the meaning intended in the passage; and that after the fear of Nero passed, the knowledge of the meaning of the number gradually faded, and had to be rediscovered long afterward. It is, however, strange, that only about a century after the writing of the Apocalypse the connection of the number with Nero was apparently unknown to Irenæus. He made several conjectures of words to fit the number.

In the later Middle Ages and afterward, the number was made to fit heresies and individual heretics. Protestants in turn found that a little ingenuity could discover a similar correspondence between the number and symbols for the papacy or names of popes. So the exchange of these expressions of regard continued. When the name is taken in Greek, the number is expressed in Greek numerals, where every letter is a numeral; but when Latin is used, only M, D, C, L, X, V, and I have numerical values.

V	I	C	A	R	I	V	S	F	I	L	I	I	D	E	I	
5	+1	+100			+1	+5			+1	+50	+1	+1	+500		+1	= 666.

This and a similar derivation from Luther's name are perhaps the most famous of these performances.

De Morgan cites a book by Rev. David Thom,* from which he quotes names, significant mottoes etc. that have been shown to spell out the number 666. He gives 18 such from the Latin and 38 from the Greek and omits those from the Hebrew. Some of these were made in jest, but many in grim earnest. He also gives a few from other sources than the book mentioned.

The number of such interpretations is so great as to destroy the claim of any. "We can not infer much from the fact that the key fits the lock, if it is a lock in which almost any key will turn." A certain interest still attaches to all such cabalistic hermeneutics, and they are not without their lesson to us, but it is not the lesson intended by the interpreter. When it comes to the use of such interpretations by one branch of the Church against another, one would prefer as less irreverent the suggestion of De Morgan, that the true explanation of the three sixes is that the interpreters are "six of one and half a dozen of the other."

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* *The Number and Names of the Apocalyptic Beasts*, part 1, 8vo, 1848. See De Morgan's *Budget of Paradoxes*, p. 402–3.

"A magic square is one divided into any number of equal squares, like a chess-board, in each of which is placed one of a series of consecutive numbers from 1 up to the square of the number of cells in a side, in such a manner that the sum of those in the same row or column and in each of the two diagonals is constant." (*Encyclopædia Britannica.*)

The term is often extended to include an assemblage of numbers not consecutive but meeting all other requirements of this definition. If every number in a magic square be multiplied by any number, q, integral or fractional, arithmetical, real or imaginary, such an assemblage is formed, and by the distributive law of multiplication its sums are each q times those in the original square.

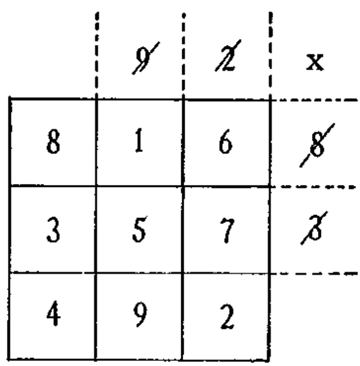

Fig. 46.

One way (De la Loubère's) of constructing any odd-number square is as follows:

1. In assigning the consecutive numbers, proceed in an oblique direction up and to the right (see 4, 5, 6 in Fig. 46).

2. When this would carry a number out of the square, write that number in the cell at the opposite end of the column or row, as shown in case of the canceled figures in the margin of Fig. 46.

3. When the application of rule 1 would place a number in a cell already occupied, write the new number, instead, in the cell beneath the one last filled. (The cell above and to the right of 3 being occupied, 4 is written beneath 3.)

4. Treat the marginal square marked x as an occupied cell, and apply rule 3.

5. Begin by putting 1 in the top cell of the middle column.

This rule will fill any square having an odd number of cells in each row and column.

The investigation of some of the properties of the simple squares just described is an interesting diversion. For example, after the 5-square and 7-square have been constructed and one is familiar with the rule, he may set himself the problem to find a formula for the sum of the numbers in each row, column or diagonal of any square. Noticing that the diagonal from lower left corner to upper right is composed of consecutive numbers, it will be easy to write the formula for the sum of that series (the required sum) if we can find the formula for the number in the lower left corner. Since the number of cells in each row or column of the squares we are considering is odd, we represent that number by the general formula for an odd number, $2n+1$. Our square, then, is a $(2n+1)$-square. If n be taken $= 1$, we have a 3-square; if $n = 2$, a 5-square; etc. Now it is seen by inspection that the number in the lower left cell is $n(2n+1)+1$, the succeeding numbers in the diagonal being $n(2n+1)+2$, $n(2n+1)+3$, etc. Summing this series to $2n+1$ terms, we have the required formula, $(2n+1)(2n^2+2n+1)$. This might be tabulated as follows (including 1 as the limiting case of a magic square):

ARBITRARY VALUES OF n (SUCCESSIVE INTEGERS)	NO. OF CELLS IN EACH ROW OR COLUMN (SUCCESSIVE ODD NUMBERS)	THE NUMBER IN THE LOWER LEFT CORNER	SUM OF THE NUMBERS IN ANY ROW, COLUMN OR DIAGONAL
n	$2n+1$	$n(2n+1)+1$	$(2n+1)(2n^2+2n+1)$
0	1	1	1
1	3	4	15
2	5	11	65
3	7	22	175
4	9	37	369
5	11	56	671
etc.	etc.		

Following is the 11-square; sum, 671:

68	81	94	107	120	1	14	27	40	53	66
80	93	106	119	11	13	26	39	52	65	67
92	105	118	10	12	25	38	51	64	77	79
104	117	9	22	24	37	50	63	76	78	91
116	8	21	23	36	49	62	75	88	90	103
7	20	33	35	48	61	74	87	89	102	115
19	32	34	47	60	73	86	99	101	114	6
31	44	46	59	72	85	98	100	113	5	18
43	45	58	71	84	97	110	112	4	17	30
55	57	70	83	96	109	111	3	16	29	42
56	69	82	95	108	121	2	15	28	41	54

There are also "geometrical magic squares," in which the *product* of the numbers in every row, column and diagonal is the same. If a number be selected as base and the numbers in an ordinary magic square be used as exponents by which to affect it, the resulting powers form a geometric square (by the first law of exponents). E. g., Take 2 as base and the numbers in the square (Fig. 46) as exponents. The resulting geometrical magic square (Fig. 47) has 2^{15} for the product of the numbers in each line.

256	2	64
8	32	128
16	512	4

Fig. 47.

The theory of magic squares in general, including even-number squares, squares with additional properties, etc., and including the extension of the idea to cubes, is given in the article "Magic Squares" in the *Encyclopædia Britannica*, together with some account of their history. See also Ball's *Recreations*; Lucas's *Récréations*, vol. 4, Cinquième Récréation, "Les Carrés magiques de Fermat"; and the comprehensive article, "A Mathematical Study of Magic Squares," by L. S. Frierson, in *The Monist* for April, 1907, p. 272–293.

The oldest manuscript on magic squares still preserved dates from the fourth or fifth century. It is by a Greek named Moscopulus. Magic squares engraved on metal or stone are said to be worn as talismans in some parts of India to this day. (*Britannica.*)

Among the most prominent of the modern philosophers who have amused themselves by perfecting the theory of magic squares is

Franklin, "the model of practical wisdom."

Domino magic squares. A pleasing diversion is the forming of magic squares with dominoes. This phase of the subject has been set forth by several writers; among them Ball,* who also mentions *coin magic squares.* The following are by Mr. Escott, who remarks: "I do not know

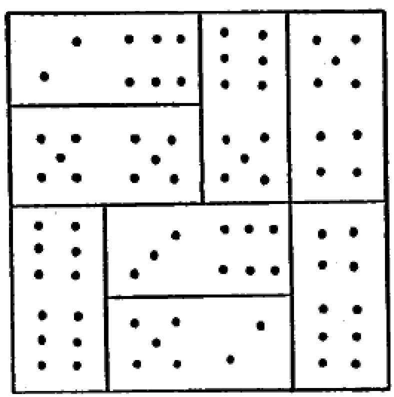

Fig. 48.

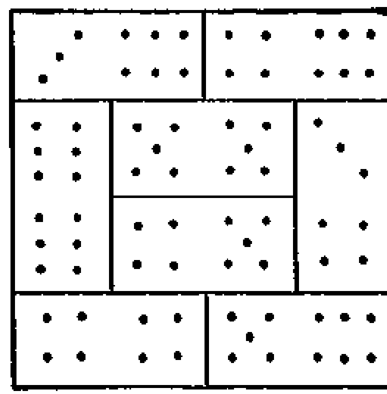

Fig. 49.

how many solutions there are. I give five [of which two are reproduced here], which I found after a few trials. In each of these magic squares the sum is the greatest possible, 19. If we subtract every number from 6, we get magic squares where the sum is the least possible, 5."

Magic hexagons.† Sum of any side of triangle = sum of vertexes of either triangle = sum of vertexes of convex hexagon = sum of vertexes of any parallelogram = 26. "There are only six solutions, of which this is one." (Fig. 50.)

Place the numbers 1 to 19 on the sides of the equilateral triangles so that the sum on every side is the same.

* *Recreations*, p. 165–6.

†From Mr. Escott, who says: "The first appeared in *Knowledge*, in 1895, and the second is due to Mr. S. Lloyd."

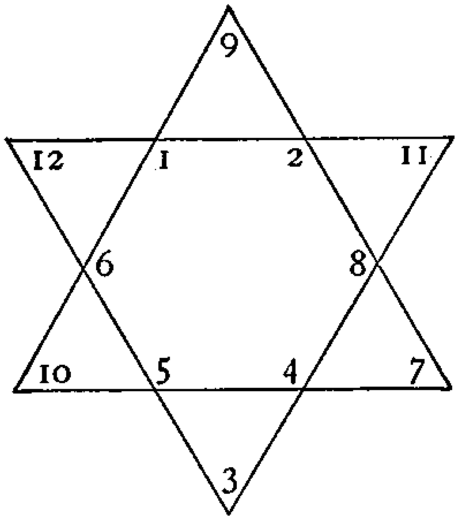

Fig. 50.

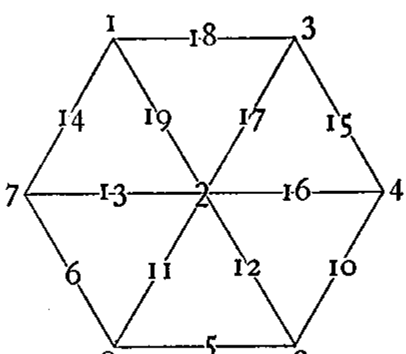

Fig. 51.

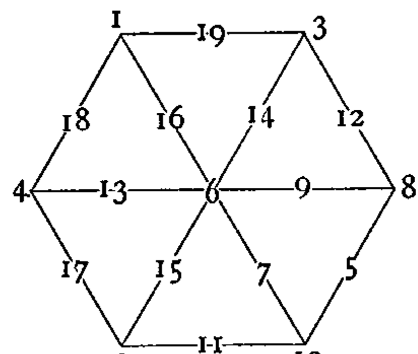

Fig. 52.

The sum on the sides of the triangles in Fig. 51 is 22. In Fig. 52 it is 23. If we subtract each of the above numbers from 20, we have solutions where the sums are respectively 38 and 37.

THE SQUARE OF GOTHAM.

(From *Teachers' Note Book*, by permission.)

The wise men of Gotham, famous for their eccentric blunders, once undertook the management of a school; they arranged their establishment in the form of a square divided into 9 rooms. The playground occupied the center, and 24 scholars the rooms around it, 3 being in each. In spite of the strictness of discipline, it was suspected that the boys were in the habit of playing truant, and it was determined to set

3	3	3
3		3
3	3	3

Fig. 53.

4	1	4
1		1
4	1	4

Fig. 54.

a strict watch. To assure themselves that all the boys were on the premises, they visited the rooms, and found 3 in each, or 9 in a row. Four boys then went out, and the wise men soon after visited the rooms, and finding 9 in each row, thought all was right. The four boys then came back, accompanied by four strangers; and the Gothamites, on their third round, finding still 9 in each row, entertained no suspicion of what had taken place. Then 4 more "chums" were admitted, but the wise men, on examining the establishment a fourth time, still found 9 in each row, and so came to an opinion that their previous suspicions had been unfounded.

Figures 53–56 show how all this was possible, as they represent

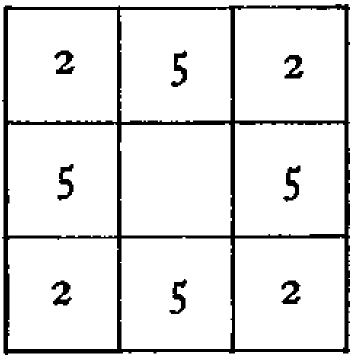

Fig. 55.

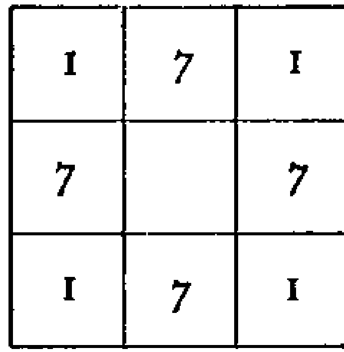

Fig. 56.

the contents of each room at the four different visits; Fig. 53, at the commencement of the watch; Fig. 54, when four had gone out; Fig. 55, when the four, accompanied by another four had returned; and Fig. 56, when four more had joined them.

A MATHEMATICAL GAME-PUZZLE.

"Place 15 checkers on the table. You are to draw (take away either 1, 2, or 3); then your opponent is to draw (take 1, 2, or 3 at his option); then you draw again; then your opponent. You are to force him to take the last one."

Solution: When your opponent makes his last draw, there must be just one checker left for him to take. Since at every draw you are limited to removing either 1, 2, or 3, you can, by your last draw, leave just 1 if, and only if, you find on the board before that draw either 2, 3, or 4. You must, therefore, after your next to the last draw, leave the board so that he can not but leave, after his next to the last draw, either 2, 3, or 4. 5 is clearly the number that you must leave at that time; since if he takes 1, he leaves 4; if 2, 3; if 3, 2. Similarly, after your next preceding draw you must leave 9; after your *next* preceding, 13; that is, *you must first draw* 2. *Then after each draw that he makes, you draw the difference between* 4 *and the number that he has just drawn*, (if he takes 1, you follow by taking 3; if he takes 2, you take 2; if he takes 3, you take 1). Four being the sum of the smallest number and the largest that may be drawn, you can always make the sum of two consecutive draws (your opponent's and yours) 4, and you can not always make it any other number.

Following would be a more general problem: Let your opponent place on the board *any* number of checkers leaving you to choose who shall first draw (1, 2, or 3 as before). Required to leave the last checker for him. Solution: If the number he places on the board is a number of the form $4n+1$, choose that he shall draw first. Then keep the number left on the board in that form by making his draw + yours = 4, until $n = 0$; that is, until there is but one left. If the number that he places on the board is not of that form; draw first and reduce it to a number that is of that form, and proceed as before.

The problem might be further generalized by varying the number that may be taken at a draw.

PUZZLE OF THE CAMELS.

There was once an Arab who had three sons. In his will he bequeathed his property, consisting of camels, to his sons, the eldest son to have one-half of them, the second son one-third, and the youngest one-ninth. The Arab died leaving 17 camels, a number not divisible by either 2, 3, or 9. As the camels could not be divided, a neighboring sheik was called in consultation.

He loaned them a camel, so that they had 18 to divide.

The first son took $\frac{1}{2}$	9
The second took $\frac{1}{3}$	6
The third took $\frac{1}{9}$	2
Total	17

They had divided equitably, and were able to return the camel that had been loaned to them.

It should be noted that $\frac{1}{2} + \frac{1}{3} + \frac{1}{9} = \frac{17}{18}$, not unity. The numbers 9, 6, 2 are in the same ratio as $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{9}$.

This is probably an imitation of the old Roman inheritance problem which may be found in Cajori's *History of Mathematics*, p. 79–80, or in his *History of Elementary Mathematics*, p. 41.

A FEW MORE OLD-TIMERS.

A man had eight gallons of wine in a keg. He wanted to divide it so as to get one-half. He had no measures but a three gallon keg, a five gallon keg and a seven gallon keg. How did he divide it? (The five gallon keg is unnecessary.)

Only one dimension on Wall street. Broker (determined to see the bright side): "Every time I bought stocks for a rise, they went down; and when sold them, they went up. Luckily they can't go sidewise."

The apple women. Two apple women had 30 apples each for sale. If the first had sold hers at the rate of 2 for 1 cent, she would have received 15 cents. If the other had sold hers at 3 for 1 cent, she would have received 10 cents. Both would have had 25 cents. But they put them all together and sold the 60 apples at 5 for 2 cents, thus getting 24 cents. What became of the other cent?

G. D. with same remainder. Given three (or more) integers, as 27, 48, 90; required to find their greatest integral divisor that will leave the same remainder.

Solution: Subtract the smallest number from each of the others. The G. C. D. of the differences is the required divisor. $48 - 27 = 21$; $90 - 27 = 63$; G. C. D. of 21 and 63 is 21. If the given numbers be divided by 21, there is a remainder of 6 in each case.

"15 Christians and 15 Turks, being at sea in one and the same ship in a terrible storm, and the pilot declaring a necessity of casting the one half of those persons into the sea, that the rest might be saved; they all agreed, that the persons to be cast away should be set out by lot after this manner, viz., the 30 persons should be placed in a round form like a ring, and then beginning to count at one of the passengers, and proceeding circularly, every ninth person should be cast into the sea, until of the 30 persons there remained only 15. The question is, how those 30 persons should be placed, that the lot might infallibly fall upon the 15 Turks and not upon any of the 15 Christians."

The early history of this problem is given by Professor Cajori in his *History of Elementary Mathematics*, p. 221–2, who also quotes mnemonic verses giving the solution: 4 Christians, then 5 Turks, then 2 Christians, etc.

The solution is really found by arranging 30 numbers or counters in a ring, or in a row to be read in circular order. Count according to the conditions of the problem, marking every ninth one "T" until 15 are marked, then mark the remaining 15 "C."

The same problem has appeared in other forms. Sometimes other classes of persons take the places of the Christians and Turks, sometimes every tenth one is lost instead of every ninth.

A FEW CATCH QUESTIONS.

What number can be divided by every other number without a remainder?

"Four-fourths exceeds three-fourths by what fractional part?" This question will usually divide a company.

Can a fraction whose numerator is less than its denominator be equal to a fraction whose numerator is greater than its denominator? If not, how can

$$\frac{-3}{+6} = \frac{+5}{-10}?$$

In the proportion

$$+6 : -3 :: -10 : +5$$

is not either extreme greater than either mean? What has become of the old rule, "greater is to less as greater is to less"?

Where is the fallacy in the following?

$$1 \text{ mile square} = 1 \text{ square mile}$$
$$\therefore 2 \text{ miles square} = 2 \text{ square miles.}$$

(Axiom: If equals be multiplied by equals, etc.)

Or in this (which is from Rebiere):

$$\text{A glass } \tfrac{1}{2} \text{ full of water} = \text{a glass } \tfrac{1}{2} \text{ empty}$$
$$\therefore \text{A glass full} = \text{a glass empty.}$$

(Axiom: If equals be multiplied.)

SEVEN-COUNTERS GAME.

Required to place seven counters on seven of the eight spots in conformity to the following rule: To place a counter, one must set out from a spot that is unoccupied and move along a straight line to the spot where the counter is to be placed.

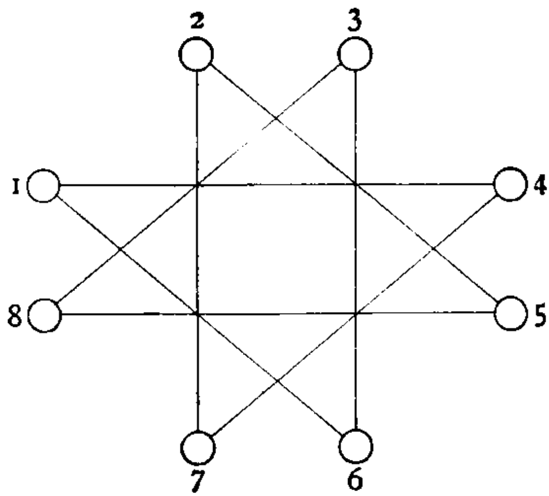

Fig. 57.

The writer remembers seeing this as a child when the game was probably new. The solution is so easy that it offered no difficulty then. A puzzle whose solution is seen by almost any one in a minute or two is hardly worth a name, and one wonders to see it in Lucas's *Récréations mathématiques* and dignified by the title "The American Game of Seven and Eight." Lucas explains that the game, invented by Knowlton, of Buffalo, N. Y., was published, in 1883, by an American journal offering at first a prize to the person who should send, within a fixed time, the solution expressed in the fewest words.

Lucas's statement of the solution is, *Take always for point of destination the preceding point of departure.* Starting, for example, from the point 4, following the line 4–1, and placing a counter at 1; the spot 4 must be the second spot of arrival. As one can reach 4 only by the line 7–4, the spot 7 will necessarily be the second point of departure; etc., the seven moves being

4–1, 7–4, 2–7, 5–2, 8–5, 3–8, 6–3.

Lucas* generalizes the game somewhat and adds other amusements with counters, less trivial than "the American game."

*Vol. 3, sixth recreation, from which the figure and description in the text are taken.

TO DETERMINE DIRECTION BY A WATCH.

Those who are familiar with this very elementary operation usually take it for granted that every one knows it. Inquiry made recently in a class of normal school students revealed the fact that but few had heard of it and not one could explain or state the method. The writer has not infrequently known well informed persons to express surprise and pleasure at hearing it.

With the face of the watch up, point the hour hand to the sun. Then the point midway between the present hour mark and XII is toward the south. E. g., at 4 o'clock, when the hour hand is held toward the sun, II is toward the south.

Or the rule may be stated thus: With the point that is midway between the present hour and XII held toward the sun, XII is toward the south. E. g., at 4 o'clock hold II toward the sun; then the line from the center of the face to the mark XII is the south line.

The reason is apparent. At 12 o'clock the sun, the hour hand and the XII mark are all toward the south. The sun and the hour hand revolve in the same direction, but the hour hand makes the complete revolution in 12 hours, the sun in 24. Hence the rule.

The errors due to holding the watch horizontal instead of in the plane of the ecliptic, and to the difference between standard time and solar time, are negligible for the purpose to which this rule is usually put.

Ball* mentions that the "rule is given by W. H. Richards, *Military Topography*, London, 1883." Being so simple and convenient, it was probably known earlier.

Professor Ball also gives (p. 356) the rule for the southern hemisphere: "If the watch is held so that the figure XII points to the sun, then the direction which bisects the angle between the hour of the day and the figure XII will point due north."

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* *Recreations*, p. 355.

MATHEMATICAL ADVICE TO A BUILDING COMMITTEE.

It will be remembered that the man who, under the pseudonym Lewis Carroll, wrote *Alice's Adventures in Wonderland* was really Rev. Charles Lutwidge Dodgson, lecturer in mathematics at Oxford. To a building committee about to erect a new school building he gave some advice that added gaiety to the deliberations. Children who have laughed at the Mock Turtle's description of his school life in the sea, as given to Alice, will recognize the same humor in these suggestions to the building committee:

"It is often impossible for students to carry on accurate mathematical calculations in close contiguity to one another, owing to their mutual interference and a tendency to general conversation. Consequently these processes require different rooms in which irrepressible conversationists, who are found to occur in every branch of society, might be carefully and permanently fixed.

"It may be sufficient for the present to enumerate the following requisites; others might be added as the funds permitted:

"A. A very large room for calculating greatest common measure. To this a small one might be attached for least common multiple; this, however, might be dispensed with.

"B. A piece of open ground for keeping roots and practising their extraction; it would be advisable to keep square roots by themselves, as their corners are apt to damage others.

"C. A room for reducing fractions to their lowest terms. This should be provided with a cellar for keeping the lowest terms when found.

"D. A large room, which might be darkened and fitted up with a magic lantern for the purpose of exhibiting circulating decimals in the act of circulation.

"E. A narrow strip of ground, railed off and carefully leveled, for testing practically whether parallel lines meet or not; for this purpose it should reach, to use the expressive language of Euclid, 'ever so far.'"

THE GOLDEN AGE OF MATHEMATICS.

"The eighteenth century was philosophic, the nineteenth scientific." Mathematics—itself "the queen of the sciences," as Gauss phrased it—is the necessary method of all exact investigation. Kepler exclaimed: "The laws of nature are but the mathematical thoughts of God." No wonder, therefore, that the nineteenth century surpassed its predecessors in extent and variety of mathematical invention and application.

One reads now of "the recent renaissance of mathematics." Strictly, there is no new birth or awakening of mathematics, for its productivity has long been continuous. Being the index of scientific progress, it must rise with the rise of civilization. That rise has been so rapid of late that, speaking comparatively, one may be justified in characterizing the present great mathematical activity as a renaissance.

"The committee appointed by the Royal Society to report on a catalogue of periodical literature estimated, in 1900, that more than 1500 memoirs on pure mathematics were now issued annually."[*]

Poets put the golden age of the race in the past. Prophets have seen that it is in the future. The recent marvelous growth of mathematics has been said to place *its* golden age in the present or the immediate future. Professor James Pierpont,[†] after summing up the mathematical achievements of the nineteenth century, exclaimed: "We who stand on the threshold of a new century can look back on an era of unparalleled progress. Looking into the future an equally bright prospect greets our eyes; on all sides fruitful fields of research invite our labor and promise easy and rich returns. Surely this is the golden age of mathematics!"

And this golden age must last as long as men interrogate nature or value precision or seek truth. "Mathematics is pre-eminently cosmopolitan and eternal."

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[*] Ball, *Hist.*, p. 455.

[†] Address before the department of mathematics of the International Congress of Arts and Science, St. Louis, Sept. 20, 1904, on "The History of Mathematics in the Nineteenth Century," *Bull. Am. Mathem. Society*, 11:3:159.

THE MOVEMENT TO MAKE MATHEMATICS TEACHING MORE CONCRETE.

With the increased mathematical production has come a movement for improved teaching. The impetus is felt in many lands. "The world-wide movement in the teaching of mathematics, in the midst of which we stand," are the recent words of a leader in this department.*

The movement is, in large part, for more concrete teaching—for a closer correlation between the mathematical subjects themselves and between the mathematics and the natural sciences, for extensive use of graphical representation, the introduction of more problems pertaining to pupils' interests and experiences, a larger use of induction and appeal to intuition at the expense of rigorous proof in the earlier years, the postponement of the more abstract topics, and the constant aim to show the useful applications.

Some of the more conservative things that are urged are what every good teacher has been doing for years. On the other hand, some of the more radical suggestions will doubtless prove impractical and be abandoned. Still the movement as a whole is healthful and full of promise.

Among American publications that are taking part in it may be mentioned the magazine *School Science and Mathematics*, which is doing much for the correlation of elementary pure and applied mathematics, the *Reports* of the various committees, the *Proceedings* of the Central Association of Science and Mathematics Teachers and similar organizations, and Dr. Young's new book, *The Teaching of Mathematics*.

The *Public School Journal* says, "The position of mathematics as a mental tonic would be strengthened," and quotes Fourier, "The deeper study of nature is the most fruitful source of mathematical study."

The movement to teach the calculus through engineering problems

*Dr. J. W. A. Young, assistant professor of the pedagogy of mathematics in the University of Chicago, in an address before the mathematical section of the Central Association of Science and Mathematics Teachers, Nov. 30, 1906.

and the like has attracted wide attention.

Some of the applications of the rudiments of descriptive geometry to drawing (mechanical, perspective etc.) are not far to seek in works on drawing.

The applications of geometry to elementary science have been given in outline. It would be well if there were available lists of the common applications in the trades. E. g., (in the carpenter's trade):

The chalk line to mark a *straight* (etymologically, *stretched*) line. Illustrating the old statement, "The straight line is the shortest distance between two points."

Putting the spirit level in two non-parallel positions on a plane surface to see whether the surface is horizontal. "A plane is determined by two intersecting straight lines."

Etc.

Perhaps most teachers of geometry have made, or induced pupils to make, some such list; but the writer is not aware that any extensive compilation is in print.

Fairly complete lists of the applications of algebra to the natural sciences may be found in the publications named above.

The new industrial arithmetic is one of the educational features of our time. There should be an arithmetic with problems drawn largely from agricultural life. The 1905 catalog of the Northern Illinois State Normal School, De Kalb, contains a valuable classified outline of child activities involving and illustrating number. Dr. Charles A. McMurry, in his *Special Method in Arithmetic*, mentions the need of "much more abundant statistical data than the arithmetics contain."

If we could have these things as teaching *material*, without the affliction of a fad for teaching mathematics *entirely* through its practical applications, it would be a boon indeed.

While rejoicing in the movement for correlation of mathematics and the other sciences, these two points should not be overlooked:

1. The sciences commonly called natural are not the only observational sciences. The field of applied mathematics is as broad as the

field of definite knowledge or investigation. Some parts of this field are specially worthy of note in this connection. The statistical sciences, the social sciences treated mathematically, the application of the methods of exact science to social measurements such as those obtained in educational psychology and the study of population, public health, economic problems etc.—these are sciences aiming at accuracy. They seek to achieve expression in natural law. They offer some of the best opportunities of applied mathematics. The recent growth in the sciences of this group has been, if possible, more marked than that of the physical sciences. Nor are they less characteristic of the spirit of our time. Indeed it has been said that the quotation beginning the preceding section should be extended so as to say, "The eighteenth century was philosophic, the nineteenth scientific, and the twentieth is to be sociologic."

The statistical sciences call for a broad acquaintance with mathematical lore which is sometimes regarded as abstract and impractical by certain critics of current mathematical curricula.* The social sciences are not studied by those who are pursuing elementary mathematical courses. It is not proposed that elementary mathematics should be correlated with them instead of with the physical sciences, or in addition thereto. But it should be remembered that use in the physical sciences is by no means the only ultimate aim which makes mathematics practical.

2. The beautiful has its place in mathematics as elsewhere. The

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*It is true that the statistical sciences are exposed to caricature, as in the story of the German statistician who announced that he had tabulated returns from the marriage records of the entire country for the year and had discovered that the number of men married that year was exactly equal to the number of women married in the same period of time! It is true that statisticians have (rarely) computed results that might have been deduced *a priori*. It is true also that some of the results of statistical science have not proved to be practical or yielded material returns. But these things might be said also of the natural sciences, whose inestimable value is everywhere recognized. The social sciences mathematically developed are to be one of the controlling factors in civilization.

prose of ordinary intercourse and of business correspondence might be held to be the most practical use to which language is put, but we should be poor indeed without the literature of imagination. Mathematics too has its triumphs of the creative imagination, its beautiful theorems, its proofs and processes whose perfection of form has made them classic. He must be a "practical" man who can see no poetry in mathematics!

Let mathematics be correlated with physical science; let it be concrete; but let the movement be understood and the subject taught in the light of the broadest educational philosophy.

THE MATHEMATICAL RECITATION AS AN EXERCISE IN PUBLIC SPEAKING.*

The value of translating from a foreign language, in broadening the vocabulary by compelling the mind to move in unfrequented paths of thought; of drawing, in quickening the appreciation of exact relations, proportion and perspective; of the natural sciences, in developing independence of thought—this is all familiar to the student of oratory; often has he been told the value of pursuing these studies for one entering his profession. But one rarely hears of the mathematical recitation as a preparation for public speaking. Yet mathematics shares with these studies their advantages, and has another in a higher degree than either of them.

Most readers will agree that a prime requisite for healthful experience in public speaking is that the attention of speaker and hearers alike be drawn wholly away from the speaker and concentrated upon his thought. In perhaps no other class-room is this so easy as in the mathematical, where the close reasoning, the rigorous demonstration, the tracing of necessary conclusions from given hypotheses, commands and secures the entire mental power of the student who is explaining, and of his classmates. In what other circumstances do students feel so instinctively that manner counts for so little and mind for so much? In what other circumstances, therefore, is a simple, unaffected, easy, graceful manner so naturally and so healthfully cultivated? Mannerisms that are mere affectation or the result of bad literary habits recede to the background and finally disappear, while those peculiarities that are the expression of personality and are inseparable from its activity continually develop, where the student frequently presents, to an audience of his intellectual peers, a connected train of reasoning.

How interesting is a recitation from this point of view! I do not recall more than two pupils reciting mathematics in an affected manner. In both cases this passed away. One of these, a lady who was previously

*Article by the writer in *New York Education*, now *American Education*, for January, 1899.

acquainted with the work done during the early part of the term, lost her mannerisms when the class took up a subject that was advance work to her, and that called out her higher powers.

The continual use of diagrams to make the meaning clear stimulates the student's power of illustration.

The effect of mathematical study on the orator in his ways of thinking is apparent—the cultivation of clear and vigorous deduction from known facts.

One could almost wish that our institutions for the teaching of the science and the art of public speaking would put over their doors the motto that Plato had over the entrance to his school of philosophy: "Let no one who is unacquainted with geometry enter here."

THE NATURE OF MATHEMATICAL REASONING.*

Why is mathematics "the exact science"? Because of its self-imposed limitations. Mathematics concerns itself, not with any problem of the nature of things in themselves, but with the simpler problems of the relations between things. Starting from certain definite assumptions, the mathematician seeks only to arrive by legitimate processes at conclusions that are surely right if the data are right; as in geometry. So the arithmetician is concerned only that the result of his computation shall be correct assuming the data to be correct; though if he is also a teacher, he is in that capacity concerned that the data of the problems set for his pupils shall correspond to actual commercial, industrial or scientific conditions of the present day.

Mathematics is usually occupied with the consideration of only one or a few of the phases of a situation. Of the many conditions involved, only a few of the most important and the most available are considered. All other variables are treated as constants. Take for illustration the "cistern problem," which as it occurs in the writings of Heron of Alexandria (c. 2d cent. B. C.) must be deemed very respectable on the score of age: given the time in which each pipe can fill a cistern separately, required the time in which they will fill it together. This assumes the flow to be constant. Other statements of the problem, in which one pipe fills while another empties, presuppose the outflow also to be constant whether the cistern is full or nearly empty; or at least the rate of outflow is taken as an average rate and treated as a constant. Or the "days-work problem" (which is only the cistern problem disguised): given the time in which each man can do a piece of work separately, required the time in which they will do it together. This assumes that the men work at the same rate whether alone or together. Some persons who have employed labor know how violent an assumption this

*vanced section of teachers institutes. For a treatment of old and new definitions of mathematics, the reader is referred to Prof. Maxime Bôcher's "The Fundamental Conceptions and Methods of Mathematics," *Bull. Am. Math. Soc.*, II:3:115–135. (Footnote text is truncated in the original.—*Transcriber.*)

is, and are prepared to defend the position of the thoughtless school boy who says, "If A can do a piece of work in 5 days which B can do in 3 days, it will take them 8 days working together," as against the answer $1\frac{7}{8}$ days, which is deemed orthodox among arithmeticians. Or, to move up to the differential calculus for an illustration: "The differentials of variables which change non-uniformly are what *would be* their corresponding increments if at the corresponding values considered the change of each became and continued uniform with respect to the same variable."*

Mathematics resembles fine art in that each abstracts some one pertinent thing, or some few things, from the mass of things and concentrates attention on the element selected. The landscape painter gives us, not every blade of grass, but only those elements that serve to bring out the meaning of the scene. With mathematics also as with fine art, this may result in a more valuable product then any that could be obtained by taking into account every element. The portrait painted by the artist does not exactly reproduce the subject as he was at any one moment of his life, yet it may be a truer representation of the man than one or all of his photographs. So it is with one of Shakespeare's historical dramas and the annals which were its "source." "The truest things are things that never happened."

Mathematics is a science of the ideal. The magnitudes of geometry exist only as mental creations, a chalk mark being but a physical aid to the mind in holding the conception of a geometric line.

The concrete is of necessity complex; only the abstract can be simple. This is why mathematics is the simplest of all studies—simplest in proportion to the mastery attained. The same standard of mastery being applied, physics is much simpler than biology: it is more mathematical. As we rise in the scale mathematically, relations become simple, until in astronomy we find the nearest approach to conformity by physical nature to a *single* mathematical law, and we see a meaning in Plato's dictum, "God geometrizes continually."

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*Taylor's *Calculus*, p. 8.

Mathematics is thinking God's thought after him. When anything is *understood*, it is found to be susceptible of mathematical statement. The vocabulary of mathematics "is the ultimate vocabulary of the material universe." The planets had for many centuries been recognized as "wanderers" among the heavenly bodies; much had come to be known about their movements; Tycho Brahe had made a series of careful observations of Mars; Kepler stated the law: Every planet moves in an elliptical orbit with the sun at one focus, and the radius vector generates equal areas in equal times. When the motion was understood, it was expressed in the language of mathematics. Gravitation waited long for a Newton to state its law. When the statement came, it was in terms of "the ultimate vocabulary": Every particle of matter in the universe attracts every other particle with a force varying directly as the masses, and inversely as the square of the distances. When any other science—say psychology—becomes as definite in its results, those results will be stated in as mathematical language. After many experiments to determine the measure of the increase of successive sensations of the same kind when the stimulus increases, and after tireless effort in the application of the "just perceptible increment" as a unit. Prof. G. T. Fechner of Leipsic announced in 1860, in his *Psychophysik*, that the sensation varies as the logarithm of the stimulus. Fechner's law has not been established by subsequent investigations; but it was the expression of definiteness in thinking, whether that thinking was correct or not, and it illustrates mathematics as the language of precision.

Mathematics, the science of the ideal, becomes the means of investigating, understanding and making known the world of the real. The complex is expressed in terms of the simple. From one point of view mathematics may be defined as the science of successive substitutions of simpler concepts for more complex—a problem in arithmetic or algebra shown to depend on previous problems and to require only the fundamental operations, the theorems of geometry shown to depend on the definitions and axioms, the unknown parts of a triangle computed from the known, the simplifications and far-reaching generalizations of the calculus, etc. It is true that we often have successive substitutions of simpler concepts in other sciences (e. g., the reduction of the forms of logical reasoning to type forms; the simplifications culminating in the formulas of chemistry; etc.) but we naturally apply the adjective *mathematical* to those phases of any science in which this method predominates. In this view also it is seen why mathematical rigor of demonstration is itself an advancing standard. "Archimedean proof" was to the Greeks a synonym for unquestionable demonstration.

If a relation between variables is stated in mathematical symbols, the statement is a formula. A formula translated into words becomes a principle if the indicative mode is used, a rule if the imperative mode.

Mathematics is "ultimate" in the generality of its reasoning. By the aid of symbols it transcends experience and the imaging power of the mind. It determines, for example, the number of diagonals of a polygon of 1000 sides to be 498500 by substitution in the easily deduced formula $n(n-3)/2$, although one never has occasion to draw a representation of a 1000-gon and could not make a distinct mental picture of its 498500 diagonals.

If there are other inhabited planets, doubtless "these all differ from one another in language, customs and laws." But one can not imagine a world in which π is not equal to 3.14159+, or e not equal to 2.71828+, though all the *symbols* for number might easily be very different.

In recent years a few "astronomers," with an enterprise that would reflect credit on an advertising bureau, have discussed in the newspapers plans for communicating with the inhabitants of Mars. What symbols could be used for such communication? Obviously those which must be common to rational beings everywhere. Accordingly it was proposed to lay out an equilateral triangle many kilometers on a side and illuminate it with powerful arc lights. If our Martian neighbors should reply with a triangle, we could then test them on other polygons. Apparently the courtesies exchanged would for some time have to be confined to the amenities of geometry.

Civilization is humanity's response to the first—not the last, or by any means the greatest—command of its Maker, "Subdue the earth and have dominion over it." And the aim of applied mathematics is "the mastery of the world quantitatively." "Science is only quantitative knowledge." Hence mathematics is an index of the advance of civilization.

The applications of mathematics have furnished the chief incentive to the investigation of pure mathematics and the best illustrations in the teaching of it; yet the mathematician must keep the abstract science in advance of the need for its application, and must even push his inquiry in directions that offer no prospect of any practical application, both from the point of view of truth for truth's sake and from a truly far-sighted utilitarian viewpoint as well. Whewell said, "If the Greeks had not cultivated conic sections, Kepler could not have superseded Ptolemy." Behind the artisan is a chemist, "behind the chemist a physicist, behind the physicist a mathematician." It was Michael Faraday who said, "There is nothing so prolific in utilities as abstractions."

ALICE IN THE WONDERLAND OF MATHEMATICS.

Years after Alice had her "Adventures in Wonderland" and "Through the Looking-glass," described by "Lewis Carroll," she went to college. She was a young woman of strong religious convictions. As she studied science and philosophy, she was often perplexed to reduce her conclusions in different lines to a system, or at least to find some analogy which would make the coexistence of the fundamental conceptions of faith and of science more thinkable. These questions have puzzled many a more learned mind than hers, but never one more earnest.

Alice developed a fondness for mathematics and elected courses in it. The professor in that department had lectured on n-dimensional space, and Alice had read E. A. Abbott's charming little book, *Flatland; a Romance of Many Dimensions, by a Square*, which had been recommended to her by an instructor.

The big daisy-chain which was to be a feature of the approaching class-day exercises was a frequent topic of conversation among the students. It was uppermost in her mind one warm day as she went to her room after a hearty luncheon and settled down in an easy chair to rest and think.

"Why!" she said, half aloud, "I was about to make a daisy-chain that hot day when I fell asleep on the bank of the brook and went to Wonderland—so long ago. That was when I was a little girl. Wouldn't it be fun to have such a dream now? If I were a child again, I'd curl up in this big chair and go to sleep this minute. 'Let's pretend.'"

So saying, and with the magic of this favorite phrase upon her, she fell into a pleasant revery. Present surroundings faded out of consciousness, and Alice was in Wonderland.

"What a long daisy-chain this is!" thought Alice. "I wonder if I'll ever come to the end of it. Maybe it hasn't any end. Circles haven't ends, you know. Perhaps it's like finding the end of a rainbow. Maybe I'm going off along one of the infinite branches of a curve."

Just then she saw an arbor-covered path leading off to one side. She turned into it; and it led her into a room—a throne-room, for there a fairy or goddess sat in state. Alice thought this being must be one of the divinities of classical mythology, but did not know which one. Approaching the throne she bowed very low and simply said, “Goddess”; whereat that personage turned graciously and said, “Welcome, Alice.” It did not seem strange to Alice that such a being should know her name.

“Would you like to go through Wonderland?”

“Oh! yes,” answered Alice eagerly.

“You should go with an attendant. I will send the court jester, who will act as guide,” said the fairy, at the same time waving a wand.

Immediately there appeared—Alice could not tell how—a courtier dressed in the fashion of the courts of the old English kings. He dropped on one knee before the fairy; then, rising quickly, bowed to Alice, addressing her as, “Your Majesty.”

It seemed pleasant to be treated with such deference, but she promptly answered, “You mistake; I am only Miss —”

Here the fairy interrupted: “Call her ‘Alice’. The name means ‘princess.’ ”

“And you may call me ‘Phool.’ ” said the courtier; “only you will please spell it with a *ph*.”

“How can I spell it when I am only speaking it?” she asked.

“*Think* the *ph*.”

“Very well,” answered Alice rather doubtfully, “but who ever heard of spelling ‘fool’ with *ph*?”

Then he smiled broadly as he replied: “I am an anti-spelling-reformer. I desire to preserve the *ph* in words in place of *f* so that one may recognize their foreign origin and derivation.”

“Y-e-s,” said Alice, “but what does *phool* come from?”

Again the fairy interrupted. Though always gracious, she seemed to prefer brevity and directness. “You will need the magic wand.”

So saying, she handed it to the jester. The moment he had the wand, the fairy vanished. And the girl and the courtier were alone in the wonderful world, and they were not strangers. They were calling each other "Alice" and "Phool." And he held the magic wand.

One flourish of that wand, and they seemed to be in a wholly different country. There were many beings, having length, but no breadth or thickness; or, rather, they were very thin in these two dimensions, and uniformly so. They were moving only in one line.

"Oh! I know!" exclaimed Alice, "This is Lineland. I read about it."

"Yes," said Phool; "if you hadn't read about it or thought about it, I couldn't have shown it to you."

Alice looked questioningly at the wand in his hand.

"It has marvelous power, indeed," he said. "To show you in this way what you have thought about, that is magic; to show you what you had never thought of, would be—"

Alice could not catch the last word. A little twitch of the wand set them down at a different point in the line, where they could get a better view of lineland. Alice thrust her hand across the line in front of one of the inhabitants. He stopped short. She withdrew it. He was amazed at the apparition: a body (or point) had suddenly appeared in his world and as suddenly vanished. Alice was interested to see how a linelander could be imprisoned between two points.

"He never thinks to go around one of the obstacles," she said.

"The line is his world," said Phool. "One never thinks of going out of the world to get around an obstacle."

"If I could communicate with him, could I teach him about a second dimension?"

"He has no apperceiving mass," said Phool laconically.

"Very good," said Alice, laughing; "surely he has no mass. Then he can get out of his narrow world only by accident?"

"Accident!" repeated Phool, affecting surprise, "I thought you were a philosopher."

"No," replied Alice, "I am only a college girl."

"But," said Phool, "you are a lover of wisdom. Isn't that what 'philosopher' means? You see I'm a stickler for etymologies."

"All right," said Alice, "I am a philosopher then. But tell me how that being can ever appreciate space outside of his world."

"He might evolve a few dimensions."

Alice stood puzzled for a minute, though she knew that Phool was jesting. Then a serious look came into his face, and he continued:

"One-dimensional beings can learn of another dimension only by the act of some being from without their world. But let us see something of a broader world."

So saying, he waved the wand, and they were in a country where the inhabitants had length and breadth, but no appreciable thickness.

Alice was delighted. "This is Flatland," she cried out. Then after a minute she said, "I thought the Flatlanders were regular geometric figures."

Phool laughed at this with so much enjoyment that Alice laughed too, though she saw nothing very funny about it.

Phool explained: "You are thinking of the Flatland where all lawyers are square, and where acuteness is a characteristic of the lower classes while obtuseness is a mark of nobility. That would, indeed, be very flat; but we spell that with a capital *F*. This is flatland with a small *f*."

Alice fell to studying the life of the two-dimension people and thinking how the world must seem to them. She reasoned that polygons, circles and all other plane figures are always seen by them as line-segments; that they can not see an angle, but can infer it; that they may be imprisoned within a quadrilateral or any other plane figure if it has a closed perimeter which they may not cross; and that if a three-dimensional being were to cross their world (surface) they could appreciate only the section of him made by that surface, so that he would appear to them to be two-dimensional but possessing miraculous powers of motion.

Alice was pleased, but curious to see more. "Let's see other dimensional worlds," she said.

"Well, the three-dimensional world, you're in all the time," said Phool, at the same time moving the wand a little and changing the scene, "and now if you will show me how to wave this wand around through a fourth dimension, we'll be in that world straightway."

"Oh! I can't," said Alice.

"Neither can I," said he.

"Can anybody?"

"They say that in four-dimensional space one can see the inside of a closed box by looking into it from a fourth dimension just as you could see the inside of a rectangle in flatland by looking down into it from above; that a knot can not be tied in that space; and that a being coming to our world from such a world would seem to us three-dimensional, as all we could see of him would be a section made by our space, and that section would be what we call a solid. He would appear to us—let us say—as human. And he would be not less human than we, nor less real, but more so; if 'real' has degrees of comparison. The flatlander who crosses the linelander's world (line) appears to the native to be like the one-dimensional beings, but possessed of miraculous powers. So also the solid in flatland: the cross-section of him is all that a flatlander is, and that is only a section, only a phase of his real self. The ability of a being of more than three dimensions to appear and disappear, as to enter or leave a room when all doors were shut, might make him seem to us like a ghost, but he would be more real and substantial than we are."

He paused, and Alice took occasion to remark:

"That is all obtained by reason; I want to see a four-dimensional world."

Then, fearing that it might not seem courteous to her guide to appear disappointed, she added:

"But I ought to have known that the wand couldn't show us anything we might wish to see; for then there would be no limit to our intelligence."

"Would unlimited intelligence mean the same thing as absolutely infinite intelligence?" Phool asked.

"That sounds to me like a conundrum," said Alice. "Is it a play on words?"

"There goes Calculus," said Phool. "I'll ask him.—Hello! Cal."

Alice looked and saw a dignified old gentleman with flowing white beard. He turned when his name was called.

While Calculus was approaching them, Phool said in a low tone to Alice: "He'll enjoy having an eager pupil like you. This will be a carnival for Calculus."

When that worthy joined them and was made acquainted with the topic of conversation, he turned to Alice and began instruction so vigorously that Phool said, by way of caution:

"Lass! Handle with care."

Alice did not like the implication that a girl could not stand as much mathematics as any one. But then she thought, "That is only a joke," and she seemed vaguely to remember having heard it somewhere before.

"If you mean," said Calculus, "to ask whether a variable that increases without limit is the same thing as absolute infinity, the answer is clearly No. A variable increasing without limit is always nearer to *zero* than to absolute *infinity*. For simplicity of illustration, compare it with the variable of uniform change, time, and suppose the variable we are considering doubles every second. Then, no matter how long it may have been increasing at this rate, it is still nearer zero than infinity."

"Please explain," said Alice.

"Well," continued Calculus, "consider its value at any moment. It is only half what it will be one second hence, and only quarter what it will be two seconds hence, when it will still be increasing. Therefore it is *now* much nearer zero than infinity. But what is true of its value at the moment under consideration is true of any, and therefore of every, moment. An infinite is always nearer to zero than to infinity."

"Is that the reason," asked Alice, "why one must say 'increases without limit' instead of 'approaches infinity as a limit'?"

"Certainly," said Calculus; "a variable can not approach infinity as a limit. Students often have to be reminded of this."

Alice had an uncomfortable feeling that the conversation was growing too personal, and gladly turned it into more speculative channels by remarking:

"I see that one could increase in wisdom forever, though that seems miraculous."

"What do you mean by miraculous?" asked Phool.

"Why—" began Alice, and hesitated.

"People who begin an answer with 'Why' are rarely able to give an answer," said Phool.

"I fear I shall not be able," said Alice. "An etymologist" (this with a sly look at Phool) "might say it means 'wonderful'; and that is what I meant when speaking about infinites. But usually one would call that miraculous which is an exception to natural law."

"We must take the young lady over to see the curve tracing," said Calculus to Phool.

"Yes, indeed!" he replied. Then, turning to Alice, "Do you enjoy fireworks?"

"Yes, thank you," said Alice, "but I can't stay till dark."

"No?" said Phool, with an interrogation. "Well, we'll have them very soon."

"Fireworks in daytime?" she asked.

But at that moment Phool made a flourish with the wand, and it was night—a clear night with no moon or star. It seemed so natural for the magic wand to accomplish things that Alice was not *very* much surprised at even this transformation. She asked:

"Did you say you were to show me curve tracing?"

"Yes," said Phool. "Perhaps you don't attend the races, but you may enjoy seeing the *traces*."

During this conversation the three had been walking, and they now came to a place where there was what appeared to be an enormous electric switchboard. A beautiful young woman was in charge.

As they approached, Calculus said to Alice, "That is Ana Lytic. You are acquainted with her, I presume."

"The name sounds familiar," said Alice, "but I don't remember to have ever seen her. I should like to meet her."

On being presented, Alice greeted her new acquaintance as 'Miss Lytic'; but that person said, in a very gracious manner:

"Nobody ever addresses me in that way. I am always called 'Ana Lytic,' except by college students. They usually call me 'Ana Lyt.' I presume they shorten my name thus because they know me so well."

In spite of the speaker's winning manner, the last clause made Alice somewhat self-conscious. Her cheeks felt very warm. She was relieved when, at that moment, Calculus said:

"This young lady would like to see some of your work."

"Some pyrotechnic curve tracing," interrupted the talkative Phool.

Calculus continued: "Please let us have an algebraic curve with a conjugate point."

Ana Lytic touched a button, and across the world of darkness (as it seemed to Alice) there flashed a sheet of light, dividing space by a luminous plane. It quickly faded, but left two rays of light perpendicular to each other, faint but apparently permanent.

"These are the axes of coordinates," explained Ana Lytic.

Then she pressed another button, and Alice saw what looked like a meteor. She watched it come from a great distance, cross the ray of light that had been called one of the axes, and go off on the other side as rapidly as it had come, always moving in the plane indicated by the vanished sheet of light. She thought of a comet; but instead of having merely a luminous tail, it left in its wake a permanent path of light. Ana Lytic had come close to Alice, and the two girls stood looking at the brilliant curve that stretched away across the darkness as far as the eye could reach.

"Isn't it beautiful!" exclaimed Alice.

Any attempt to represent on paper what she saw must be poor and inadequate. Figure 58 is such an attempt.

Suddenly she exclaimed: "What is that *point* of light?" indicating by gesture a bright point situated as shown in the figure by P.

"That is a point of the curve," said Ana Lytic.

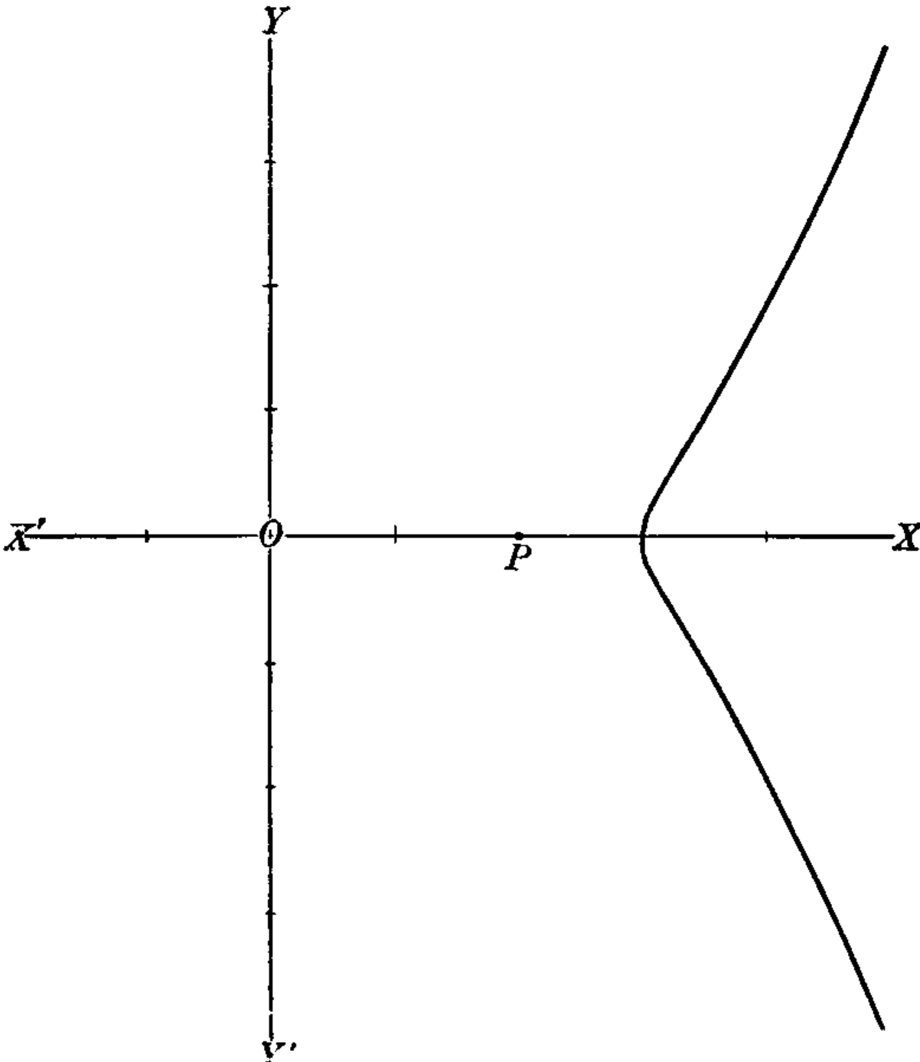

Fig. 58.

"But it is away from all the rest of it," objected Alice.

Going over to her apparatus and taking something—Alice could not see what—Ana Lytic began to write on what, in the darkness, might surely be called a blackboard. The characters were of the usual size of writing on school boards, but they were characters of light and could be plainly read in the night. This is what she wrote:

$$y^2 = (x-2)^2(x-3).$$

Stepping back, she said: "That is the equation of the curve."

Alice expressed her admiration at seeing the equation before her and its graph stretching across the world in a line of light.

"I never imagined coordinate geometry could be so beautiful," she said.

"This is throwing light on the subject for you," said Phool.

"The point about which you asked," said Ana Lytic to Alice, "is the point $(2, 0)$. You see that it satisfies the equation. It is a point of the graph."

Alice now noticed that units of length were marked off on the dimly seen axes by slightly more brilliant points of light. Thus she easily read the coordinates of the point.

"Yes," she said, "I see that; but it seems strange that it should be off away from the rest."

"Yes," said Calculus, who had been listening all the time. "One expects the curve to be continuous. Continuity is the message of modern scientific thought. This point seems to break that law—to be 'miraculous,' as you defined the term a few minutes ago. If all observed instances but one have some visible connection, we are inclined to call that one miraculous and the rest natural. As only that seems wonderful which is unusual, the miraculous in mathematics would be only an isolated case."

"I thank you," said Alice warmly. "That is the way I should like to have been able to say it. An isolated case is perplexing to me. I like to think that there is a universal reign of law."

"*Evidently*," said Phool, "here is an exception. It is *obvious* that there are several alternatives, such as, for example, that the point is not on the graph, that the graph has an isolated point, *and so forth*."

Calculus, Ana Lytic and Phool all laughed at this. To Alice's inquiry, Phool explained:

"We often say 'evidently' or 'obviously' when we can't give a reason, and we conclude a list with 'and so forth' when we can't think of another item."

Alice felt the remark might have been aimed at her. Still she had not used either of these expressions in this conversation, and Phool had made the remark in a general way as if he were satirizing the foibles of the entire human race. Moreover, if she felt inclined to resent it as an impertinent criticism from a self-constituted teacher, she remembered that it was only the jest of a jester and treated it merely as an interruption.

"Tell me about the isolated point," she said to Calculus.

He proceeded in a teacher-like way, which seemed appropriate in him.

Calculus. For $x = 2$ in this equation, $y = 0$. For any other value of x less than 3, what would y be?

Alice. An imaginary.

Calculus. And what is the geometric representation of an imaginary number?

Alice. A line whose length is given by the absolute, or arithmetic, value of the imaginary and whose direction is perpendicular to that which represents positives and negatives.

Calculus. Good. Then—

Alice (bounding with delight at the discovery). Oh! I see! I see! There must be points of the graph outside of the plane.

Calculus. Yes, there are imaginary branches, and perhaps Ana Lytic will be good enough to show you now.

The dotted line QPQ, if revolved 90° about XX′ as axis, remaining in that position in plane perpendicular to paper, would be the "imaginary part" of the graph.

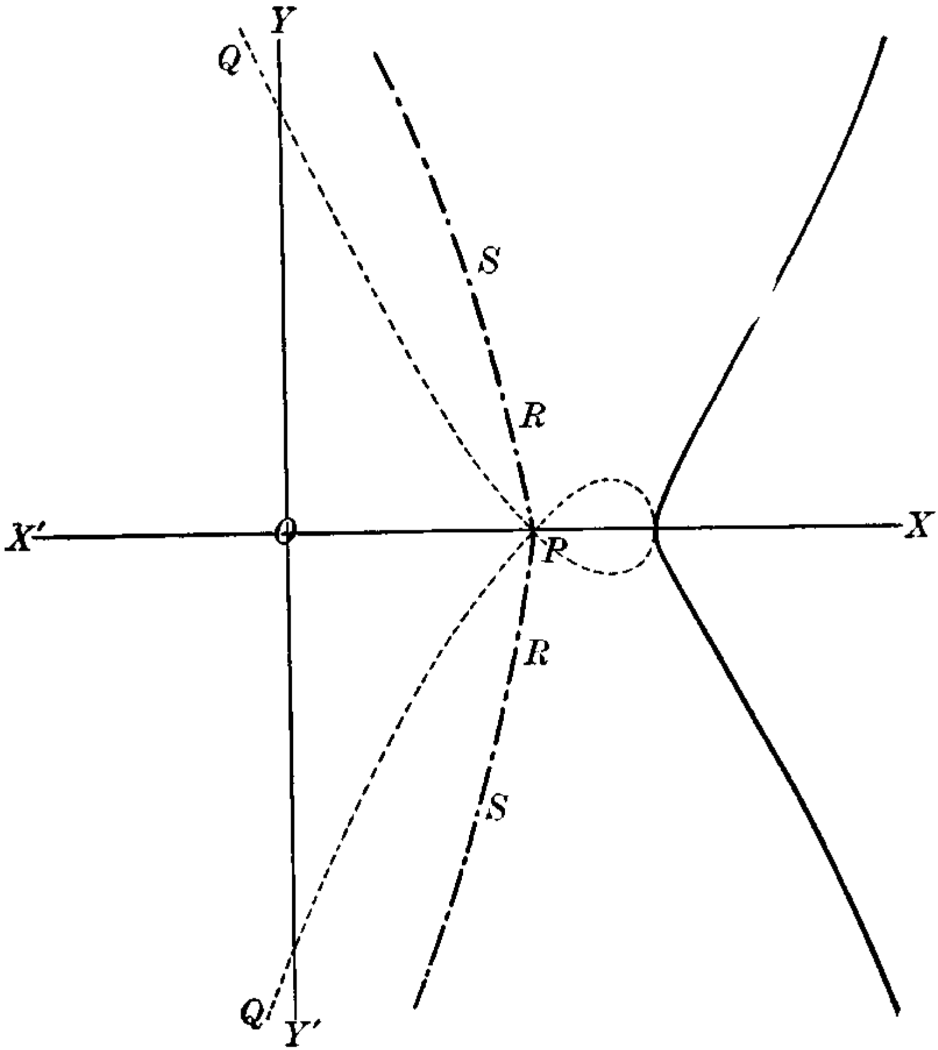

Fig. 59.

The dot-and-dash line SRPRS represents the projection on the plane of the paper of the two "complex parts." At P each branch is in the plane of the paper, at each point R one branch is about 0.7 from the plane each side of the paper, at S each branch is 1.5 from the plane, etc.

That young lady touched something on her magic switchboard, and another brilliant curve stretched across the heavens. The plane determined by it was perpendicular to the plane previously shown. (The dotted line in Fig. 59 represents in a prosaic way what Alice saw.)

"O, I see!" exclaimed Alice. "That point is not isolated. It is the point in which this 'imaginary' branch, which is as *real* as any, pierces the plane of the two axes."

"Now," said Calculus, "if instead of substituting real values for x and solving the equation for y, you were to substitute real numbers for y and solve for x, you would, in general, obtain for each value of y one real and two complex numbers as the values of x. The curve through all the points with complex abscissas is neither in the plane of the axes nor in a plane perpendicular to it. But you shall see."

(The dot-and-dash line in Fig. 59 represents these branches.)

When Ana Lytic made the proper connection at the switchboard, these branches of the curve also stood out in lines of light.

Alice was more deeply moved than ever. There was a note of deep satisfaction in her voice as she said:

"The point that troubled me because of its isolation is a point common to several branches of the curve."

"The supernatural is more natural than anything else," said Phool.

"The miraculous," thought Alice, "is only a special case of a higher law. We fail to understand things because they are connected with that which is out of our plane."

She added aloud: "This I should call the *miracle curve*."

"Yet there is nothing exceptional about this curve," said Calculus. "Any algebraic curve with a conjugate point has similar properties."

Then Calculus said something to Ana Lytic—Alice could not hear what—and Ana Lytic was just touching something on the switchboard when there was a crash of thunder. Alice gave a start and awoke to find herself in her own room at midday, and to realize that the slamming of a door in the corridor had been the thunder that terminated her dream.

She sat up in the big chair and, with the motion that had been characteristic of her as a little girl, gave "that queer little toss of her head, to keep back the wandering hair that *would* always get into her eyes," and said to herself:

"There aren't any curves of light across the sky at all! And worlds of one or two dimensions exist only in the mind. They are abstractions. But at least they are thinkable. I'm glad I had the dream. Imagination *is* a magic wand.—The future life will be a *real* wonderland, and—"

Then the ringing of a bell reminded her that it was time to start for an afternoon lecture, and she heard some of her classmates in the corridor calling to her, "Come, Alice."

Mathematical recreations. The Ahmes papyrus, oldest mathematical work in existence, has a problem which Cantor interprets as one proposed for amusement. At which Cajori remarks:* "If the above interpretations are correct, it looks as if 'mathematical recreations' were indulged in by scholars forty centuries ago."

The collection of "Problems for Quickening the Mind" Cantor thinks was by Alcuin (735–804). Cajori's interesting comment† is: "It has been remarked that the proneness to propound jocular questions is truly Anglo-Saxon, and that Alcuin was particularly noted in this respect. Of interest is the title which the collection bears: 'Problems for Quickening the Mind.' Do not these words bear testimony to the fact that even in the darkness of the Middle Ages the mind-developing power of mathematics was recognized?"

Later many collections of mathematical recreations were published, and many arithmetics contained some of the recreations. Their popularity is noticeable in England and Germany in the seventeenth and eighteenth centuries.‡

A good bibliography of mathematical recreations is given by Lucas.§ There are 16 titles from the sixteenth century, 33 from the seventeenth, 38 from the eighteenth, and 100 from the nineteenth century, the latest

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* *Hist. of Elem. Math.*, p. 24.

† *Id.*, p. 113–4.

‡ A book entitled *Rara Arithmetica* by Prof. David Eugene Smith, is to be published by Ginn & Co. the coming summer or fall (1907). It will contain six or seven hundred pages and have three hundred illustrations, presenting graphically the most interesting facts in the history of arithmetic. Its author's reputation in this field insures the book an immediate place among the classics of mathematical history.

§ I:237–248. Extensive as his list is, it is professedly restricted in scope. He says, Nous donnons ci-après, suivant l'ordre chronologique, l'indication des principaux livres, mémoires, extraits de correspondance, qui ont été publiés sur l'Arithmétique de position et sur la Géométrie de situation. Nous avons surtout choisi les documents qui se rapportent aux sujets que nous avons traités ou que nous traiterons ultérieurement.

date being 1890. Young (p. 173–4) gives a list of 20 titles, mostly recent, in no case duplicating those of Lucas's list (except where mentioning a later edition). This gives a total of over two hundred titles. Now turn to two other collections, and we find the list greatly extended. Ahrens' *Mathematische Unterhaltungen* (1900) has a bibliography of 330 titles, including nearly all those given by Lucas. Fourrey's *Curiositées Géométriques* (1907) has the most recent bibliography. It is extensive in itself and mostly supplementary to the lists by Lucas and Ahrens.

In all the vast number of published mathematical recreations, the present writer does not know of a book covering the subject in general which was written and published in America. We seem to have taken our mathematics very seriously on this side of the Atlantic.

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The Open Court, January 1907, p. 184; February, p. 179; March, p. 54, 57; April, p. 90; May, p. 122, 131, 167, 103; June, p. 62, 64; July, p. 142, 143.

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Some of the articles have been altered slightly since their publication in periodical form.

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GENERAL INDEX.

Note: 43f means page 43 and the page or pages immediately following. 149n means note at bottom of page 149. References given in the Bibliographic Index (preceding pages) are not (except in rare instances) repeated here.